VOLUME XIX SECOND SERIES NUMBER 1

THE PHYSICAL REVIEW

A JOURNAL OF EXPERIMENTAL AND THEORETICAL PHYSICS

CONDUCTED BY

THE AMERICAN PHYSICAL SOCIETY

BOARD OF EDITORS

FREDERICK BEDELL, Managing Editor

E. L. NICHOLS	G. S. FULCHER	E. P. ADAMS
C. M. SPARROW	F. A. SAUNDERS	A. J. DEMPSTER
W. F. G. SWANN	O. M. STEWART	A. L. HUGHES

JANUARY, 1922

Theory of Longitudinal Vibrations of Viscous Rods. W. G. CADY 1
The Effect of Field Direction on Magneto-Resistance. C. W. HEAPS 7
On the Principle of Combination and Stokes' Law in the X-Ray Series. D. COSTER 20
The Mobility of Electrons in Pure Nitrogen. LEONARD B. LOEB 44
An Experiment on Electromagnetic Induction and Relative Motion. W. F. G. SWANN 56
A New Tone Generator. C. W. HEWLETT 59
Measurement of Time Intervals with a Galvanometer having Fluxmeter Characteristics. PAUL E. KLOPSTEG 61
The Effect of Damping on the Width of X-Ray Spectrum Lines. G. E. M. JAUNCEY 64
The Width of X-Ray Spectrum Lines. ARTHUR H. COMPTON 68

PUBLISHED BY
THE AMERICAN PHYSICAL SOCIETY
LANCASTER, PA., AND ITHACA, N. Y.
Entered at the postoffice at Lancaster, Pa., as second class matter

OFFICERS OF THE SOCIETY

THORNDIKE LUFKIN, Harvard University, Cambridge, Mass.
President

C. E. MENDENHALL, Univ. of Wisconsin, Madison, Wis.
First Vice-President

D. C. MILLER, Case School of Applied Science, Cleveland, Ohio
Second Vice-President

G. B. PEGRAM, Columbia University, New York City
Secretary

E. BUCHER, Cornell University, Ithaca, N. Y.
Treasurer

EDWARD C. E. P. LEWIS, University of California, Berkeley, Cal.
Editor

The Council of the Society consists of the president, vice-presidents, secretary, treasurer, editor, all living past presidents and eight elected members as follows:

Elected Members: A. A. Michelson, A. G. Webster, Carl Barus, E. L. Nichols, Henry Crew, W. F. Magie, Ernest Merritt, E. A. Millikan, F. E. Nipher.

Elected Members: G. K. Burgess, J. C. McLennan, E. B. Rosa, Miss Sosman, E. P. Thompson, H. M. Randall, H. G. Gale, D. F. Webster.

A complete list of past officers is given on a back page of the Register.

THE PHYSICAL REVIEW

Two Volumes Annually

Subscriptions, throughout the Postal Union, $5.00, in advance; Single Copies, $1.25. General Index 1893–1920 (limited June, 1921) $4.00, postage extra. Back Volumes (either series) $4.50 per volume, carriage extra. Back Numbers (either series) $1.75 a copy, postage extra. Discount to subscription agencies only; ten per cent.

Remittance.—When remittance accompanies the order, the Index will be sent prepaid for $4.00 and single back numbers will be sent prepaid for $1.75. Back numbers will only be sent when they can be furnished without breaking a volume, and back volumes when they can be furnished without breaking a continuous series. Orders that can not be completely filled will be filled so far as stock permits, unless otherwise specified in the order. Money will be refunded for any defective copies returned. No bound volumes can be supplied.

Binding.—The Review offers to purchasers, at $1.50 a copy, a limited number of Vol. I, First Series, and Nos. 1, Vol. IX, February 1917, Second Series.

Subscription certificates will be furnished with bills, when so ordered. Six twenty-five cents additional copies of bills, on forms furnished by customer, will be furnished for five cents each. Duplicates will be supplied for fifteen cents each.

Members of the Physical Society may obtain back numbers at a discount of 10 per cent. A proper statement of membership accompanies order. When a member's name has been removed from the mailing list of the Review on account of non-payment of dues, it will be restored to the list when dues are paid; any numbers that he has failed to receive in the interim will be furnished without cost upon application to the Review. *All complaints for failure to receive the Review should be addressed to the Editor and not to any officer of the Society.*

First Series. From July 1, 1893, to January 1, 1913, the Review was conducted by Edward L. Nichols, Ernest Merritt and Frederick Bedell, with the financial support of Cornell University. During this period thirty-five volumes were issued, constituting the First Series. Four volumes were issued annually; subsequent volumes, semi-annually.

Second Series. On January first, 1913, the Review was transferred to the American Physical Society. A second series was then begun, the January, 1913, number being dated Volume I, Number 1, Second Series. (An announcement of this transfer will be found on the first page of the new series.) Semi-annual volumes begin in January and July.

For advertising rates, address THE PHYSICAL REVIEW, *Ithaca, N. Y.* Changes of copy for standing advertisements should be sent direct to the printers, THE NEW ERA PRINTING CO., *Lancaster, Pa. Old copy is used until new copy is received.*

Contributors to the Review are referred to page two of this advertisement for further information. Permission is given to republish any articles appearing in the Review, provided credit is given.

Address Correspondence to

THE PHYSICAL REVIEW, ITHACA, N. Y.

CONTENTS OF VOL. XIX., SECOND SERIES

JANUARY, 1922

Theory of Longitudinal Vibrations of Viscous Rods. W. G. CADY 1
The Effect of Field Direction on Magneto-Resistance. C. W. HEAPS 7
On the Principle of Combination and Stokes' Law in the X-Ray Series. D. COSTER ... 20
The Mobility of Electrons in Pure Nitrogen. LEONARD B. LOEB 24
An Experiment on Electromagnetic Induction and Relative Motion. W. F. G. SWANN. 38
A New Tone Generator. C. W. HEWLETT 52
Measurement of Time Intervals with a Galvanometer having Fluxmeter Characteristics. PAUL E. KLOPSTEG 61
The Effect of Damping on the Width of X-Ray Spectrum Lines. G. E. M. JAUNCEY .. 64
The Width of X-Ray Spectrum Lines. ARTHUR H. COMPTON 68

FEBRUARY, 1922

On the Gamma-Ray Action of Extensive Flat Radium Preparations at Different Distances with and without Absorbing Materials. VICTOR F. HESS 73
Physical Characteristics of Normal and Abnormal Ears. JOHN P. MINTON 80
The Effect of Oxygen and Hydrogen on the Emission of Electrons from Hot Platinum. LAURICE L. LOCKROW 97
The Electron Theory of Metals in the Light of New Experimental Data. P. W. BRIDGMAN 114
A Graphical Study of the Stability Relations of Atom Nuclei. WILLIAM D. HARKINS AND S. L. MADORSKY 135
Time and Pressure Measurements in the Corona. CHARLES S. FAZEL 157
On the Pressure in the Corona Discharge. JAKOB KUNZ 165
Behavior of Free Electrons Toward Gas Molecules. H. B. WAHLIN 173
Note on the Gravitational Acceleration of Bismuth. HAROLD H. POTTER 188

MARCH, 1922

The Measurement of the Resonance, Radiation, and Ionization Potentials of Several Gases and Vapors. PAUL EDWARD BOUCHER 189
Multiple Valency in the Ionization by Alpha Rays. T. R. WILKINS 210
Analysis of the Energy Distribution in Speech. I. B. CRANDALL AND D. MACKENZIE .. 221
Note on the Distribution of Range of Recoil Atoms. K. T. COMPTON 233
The Variation with the Temperature of the Thermal Conductivity of Cast Iron. ELMER E. HALL 237
Proceedings of the American Physical Society 241

Minutes of the One hundred and eleventh Meeting; Molecular Models: Benzene, *Jared Kirkland Morse;* The Mobilities of Electrons in Pure N_2, *Leonard B. Loeb;* On the Pressure Increase in the Corona Discharge, *Jakob Kunz;* The Vibration of Wires in the Corona, *Charles S. Fazel;* The Path of a Rigid Electron Moving in a Magnetic Field of Constant Strength Rotating with Constant Angular Velocity, *E. O. Hulburt;* The Effect of Ageing on the Secondary Electron Emission from Copper Surfaces, *L. E. McAllister;* Carbon Dioxide Band Spectra in the near Infra-red, *E. F. Barker;* Recent Determinations of the Susceptibilities of Oxygen and Nitric Oxide and the Magneton, *E. C. Fritts;* The Growth and Decay of Photo-Thermionic Currents from Oxide-Coated Filaments, *H. D. Arnold and Herbert E.*

417476

Ives; The Absolute Sizes of Certain Monovalent and Bivalent Ions, *Wheeler P. Davey;* On the Universal Distance of the Order of 10^{-8} cm. between the Centers of the Nearest Atoms in Solids, *Albert C. Crehore;* The Classification and the Prediction of Isotopes, *William D. Harkins;* The Nature of Charcoal "Sorption," *H. Horton Sheldon;* The Scattering of Electrons by Nickel, *C. Davisson and C. H. Kunsman;* Width of Spectral Lines of Helium as a Function of Pressure in the Source, *Lloyd W. Taylor;* On the Vacuum Spark Spectrum of Silicon, *R. A. Sawyer and R. F. Paton;* The Sparking Potential in Argon at Reduced Pressures, *E. R. Stoekle;* "The Action of Light on a Photographic Film," *Jay W. Woodrow;* The Instructional Value of Certain Types of Motion Pictures, *Harvey B. Lemon;* Sensibility of the Ear to Small Differences in Intensity and Frequency, *V. O. Knudsen;* The Efficiency of Artificial Aids to Hearing, *Paul E. Sabine;* Doppler's Principle Illustrated by Ripple Waves, *F. R. Watson and W. B. Worsham;* Einstein's Relativity and Gravitation, *John Millis;* Some Aspects of the Theory of Relativity, *A. C. Lunn;* Atomic Constants and Dimensional Invariants, *A. C. Lunn;* Measurements of the Amount of Scattered Homogeneous X-Rays of Wave-Length 0.712 Å. per Gram of Carbon, *C. W. Hewlett;* The Spectrum of Secondary X-Rays, *Arthur H. Compton;* Oscillations of Temperature of an Incandescent Filament, and the Specific Heat of Tungsten, *K. K. Smith and P. W. Bigler;* Effect of Tension on Thermoelectromotive Forces by Magnetization, *Alpheus W. Smith;* Positive-Ray Analysis of Lithium and Zinc, *A. J. Dempster;* A Convenient Contactor for Small Currents, *Frederick J. Schlink;* The Dielectric Constant of Mica, *J. R. Weeks, Jr.;* A Photomicrographic Study of a Series of Drawn Tungsten Wires, *L. P. Sieg;* A Simplified Method of Correcting for the Decrement of a Decremeter, *R. R. Ramsey;* A New Visual Null Method for Conductivity Determinations, *Fabian M. Kannenstine and Esme E. Rosaire;* Phase Relations in Coupled Circuits, *N. H. Williams;* The Change of Mobility of the Positive Ions with Age in Oxygen and Nitrogen, *Henry A. Erikson;* Spherical Aberration in Thin Lenses, *T. Townsend Smith;* A Low Resistance Connection with a Revolving Shaft, *A. P. Carman;* The Effect of Pressure and Gas Content on the Action of Vacuum-Tube Detectors, *H. A. Brown and Chas. T. Knipp;* Some Remarks on Electromagnetic Induction, *S. J. Barnett;* The Existence of Small Ions of Very High Mobilities, *Oswald Blackwood;* Effect of Lunar Gravity upon a Quartz Thread Balance, *R. C. Hartsough;* On the Electrical Properties of Illium, *Chas. T. Knipp and J. L. Hall.*

APRIL, 1922

The Effect of Tension on the Change of Thermoelectromotive Forces by Magnetization. ALPHEUS W. SMITH 285

The Luminescence of Incandescent Solids. E. L. NICHOLS AND H. L. HOWES 300

The Dielectric Constant of Mica. J. R. WEEKS, JR. 319

A New Gyroscopic Construction in Simple Vectors. WILL C. BAKER 323

The Path of a Rigid Electron which moves in a Magnetic Field of Constant Strength Rotating with Constant Angular Velocity. E. O. HULBURT 329

The Thermophone. EDWARD C. WENTE 333

Atomic Radii. I. MAURICE L. HUGGINS 346

The Crystal Structures of Aragonite ($CaCO_3$) and Related Minerals. MAURICE L. HUGGINS 354

The Crystal Structure of Quartz. MAURICE L. HUGGINS 363

The Crystal Structures of Marcasite (FeS_2), Arsenopyrite (FeAsS) and Loellingite ($FeAs_2$). MAURICE L. HUGGINS 369

Proceedings of the American Physical Society 374

Minutes of the 112th Regular Meeting; The State of Polarization of Light Diffracted at Large Angles from Straight Edges of Different Material, *J. M. Kuehne;* An Experimental Method for Determining Quantitatively under given Conditions,

the Relative Influence of Radiation and Convection in the Transfer of Heat through Moving Air, *L. H. Nichols;* Geometrical Proof for the Wadsworth Constant Deviation System, *R. C. Gibbs and J. R. Collins;* A Piezo-electric Method for Generating Electric Oscillations of Constant Frequency, *W. G. Cady;* The Dissipation of Heat in Air from Plain and Blackened Brass Surfaces for Moderate Temperature Differences, *T. S. Taylor;* Radiation Equilibrium in an Opaque Uniformly Heated Solid and Duane's Hypothesis as to its Origin, *A. G. Worthing;* An Increase in Density in Thin Films, *Elizabeth R. Laird;* Electrical Resistance of a Rotating Coil, *A. P. Carman;* The Departure from Ohm's Law in Gold and Silver at High Current Densities, *P. W. Bridgman;* Mutual Impedances of Grounded Circuits, *George A. Campbell;* Contact Electromotive Forces due to Films on Metallic Surfaces, *A. E. Hennings;* Resistance and Thermoelectric Power of Germanium, *C. C. Bidwell;* The Production of an E.M.F. on Closed Circuit, by a Light Effect on Argentite, *H. Horton Sheldon and Paul H. Heiger;* Demonstration of the Wind in the Corona Discharge, *Jakob Kunz;* Increased Ionization over Solar Faculæ, *Charles E. St. John;* A Note on Missing Spectra, *A. S. Eve;* The Ultraviolet Absorption of Anthracene and Benzene, *J. M. Hyatt;* The Ultraviolet Transmission of Boiled Absolute Ethyl Alcohol, *W. R. Orndorff, R. C. Gibbs and M. Scott;* The Absorption by Water Vapor of Radiation of Wave-Lengths near 6 μ, *W. W. Sleator and E. R. Phelps;* The Infra-red Absorption Spectra of Diatomic Gases and their Ionization Potentials, *Edwin C. Kemble;* The Effect of Certain Dissolved Substances on the Infra-red Absorption of Water, *J. R. Collins;* The Absorption Sepctrum of Hydrogen Chloride at 200° K., *B. J. Spence and C. Holley;* Two New Lines in the Aluminum Spectrum and their Possible Series Relations, *G. D. Shallenberger;* The Aspherical Nucleus Theory Applied to the Principal Series of Helium, *Ludwik Silberstein;* A Physical Explanation of the Action of the New Singing Tube, *Chas. T. Knipp and Jakob Kunz;* Theory of Acoustic Wave Filters: The Limiting Frequencies of Transmission, *G. W. Stewart;* Diffraction Effects in Sound Absorption Measurements, *Paul E. Sabine;* Analysis of the Energy Distribution in Speech, *I. B. Crandall and Donald MacKenzie;* The Separation of the Element Chlorine into Isotopes (Isotopic Atomic Species), *William D. Harkins and Anson Hayes;* On the Determination of Surface Thermal Conductivities, *Letha A. Smith;* The Causes of Reaction Expansions in Amalgams, *Arthur W. Gray;* The Porous Plug Coefficient for Air, *J. R. Roebuck;* Ether-Drift Experiments at Mount Wilson Solar Observatory, *Dayton C. Miller;* Photo-Elasticity or Optical Investigations of Stress Distribution in Solid Bodies, *Paul Heymans;* Experiments on the Sign of the Electric Charge Assumed by a Metal Immersed in a Liquid. I., *Richard D. Kleeman and William Frederickson;* The Analysis of Simple Periodic Curves by a Projection Method with Special References to Estuary Tidal Problems, *E. V. Henry;* A Galvanometer Method for Producing at a Distance a Magnified Record of a Mechanical Motion, *F. Wenner;* The Influence of Small Variations in the Outlet of a Smooth Brass Tube on the Distribution of Static Pressure when Air is flowing through the Tube, *T. S. Taylor;* Cohesion and Adhesion, *Wilson Taylor;* Progress of Work on Selenium Crystals, *LeRoy D. Weld;* The Transmitted Effect in Selenium Crystals, *J. C. Pomeroy;* Further Experiments on Critical Potentials in Hydrogen, *F. J. Mohler, Paul D. Foote and E. H. Kurth;* Photo-electric Phenomena in Coated-Filament Audion Bulbs, *R. C. Gibbs and Edna L. Meacham;* The Information Service of the National Research Council, *Robert M. Yerkes;* Power Factors in Radio Circuits, *N. H. Williams;* "J" Radiation: A Summary, *F. K. Richtmyer;* The Dilemma of the Helium Atom, *John H. Van Vleck;* Moment of Momentum of Magnetic Electrons, *E. H. Kennard;* Theory of Ionization by Cumulative Action, *K. T. Compton;* A Significant Exception to the Principle of Selection, *Paul D. Foote, F. L. Mohler and W. F. Meggers;* The Charge on the Alpha Particle, *J. E. Shrader;* Radiation from a Group of Electrons, *Leigh Page;* A Magnetic Fall of Potential Method for Testing Short Bars of Iron,

Arthur Whitmore Smith; A New Method for the Determination of the Magnetic Susceptibilities of Gases, *Jakob Kunz and E. C. Fritts;* A Sine Galvanometer for Determining in Absolute Measure the Horizontal Intensity of the Earth's Magnetic Field, *S. J. Barnett;* Results of Comparisons of Instruments for Measuring the Earth's Magnetic Elements, *Louis A. Bauer and J. A. Fleming;* Further Results of Line Integrals of the Earth's Magnetic Force, *Louis A. Bauer and W. J. Peters;* Characteristic X-Rays from Light Elements, *A. Ll. Hughes;* Energy Relations between X- and β-Rays, *J. A. Gray;* Positive-Ray Analysis of Zinc and Calcium, *A. J. Dempster;* The Effect of Dissolved Substances on the Deposition of Colloidal Particles from a Solution by Means of an Electric Current, *Richard D. Kleeman;* The Effective Range of β-Rays, *J. A. Gray and A. V. Douglas;* On the High Frequency Rays in the γ-ray Spectrum of Radium B and C, *Alois F. Kovarik;* The Beginning of the K and L Series of X-Rays, *F. L. Mohler and Paul D. Foote;* Secondary X-Rays from Crystals, *G. E. M. Jauncey;* Brightness of Tungsten, *W. E. Forsythe;* The Luminescence of Solidifying Antimony, *E. Karrer;* The Thermionic Work Function of Tungsten, *C. Davisson and L. H. Germer;* A Spectroscopic Method for Determining the Effective Temperature of Absorbing and Emitting Molecules, *Raymond T. Birge;* The Excitation of the Enhanced Spectra of Sodium and Potassium in a Low Voltage Arc, *Paul D. Foote, W. F. Meggers, and F. L. Mohler;* The Significance of the $\frac{1}{4}$ Terms in Spectral Series Formulæ, *Paul D. Foote and F. L. Mohler;* A Method for Increasing the Carrying Capacity of a Rheostat, *W. E. Forsythe;* A Transition or Adsorption Layer Theory of the E.M.F. of the Voltaic Cell, *Richard D. Kleeman;* Quantum Theory of Photographic Exposure, *Ludwik Silberstein;* The Physical Constituents of the Atmosphere of Venus, *Chas. E. St. John and Seth B. Nicholson;* The Separation of Mercury into Isotopes, *William D. Harkins and R. S. Mullikan;* The Crystal Structure of Mercury, *L. W. McKeehan and P. P. Cioffi.*

MAY, 1922

Resistance and Thermo-electric Power of Metallic Germanium. C. C. BIDWELL..... 447
Magnetostriction with Small Magnetizing Fields. JOHN R. HOBBIE, JR..... 456
A Note on the Pressure of Radiation on Transparent Dielectrics. SATYENDRA RAY... 467
The Electrodeless Discharge in Certain Vapors. J. K. ROBERTSON..... 470
Piezo-electric Activity of Rochelle Salt under Various Conditions. J. VALASEK..... 478
Minimum Sound Energy for Audition for Tones of High Frequency. C. E. LANE..... 492
The Sensitivity and Precision of the Electrostatic Transmitter for Measuring Sound Intensities. F. C. WENTE..... 498
Reflection of X-Rays from Rock Salt. BERGEN DAVIS AND W. M. STEMPEL..... 504
Solubility of CO_2 and N_2O in Certain Solvents. WILLIAM KUNERTH..... 512
Proceedings of the American Physical Society..... 525

JUNE, 1922

The Frequency-Sensitivity of Normal Ears. H. FLETCHER AND R. L. WEGEL..... 553
On the Characteristics of Highly Sensitive Discharge Points. JOHN ZELENY..... 566
On Reflection from a Moving Mirror and the Michelson-Morley Experiment. E. H. KENNARD AND D. E. RICHMOND..... 572
Problems of Quantum Theory in the Light of the Theory of Perturbations. PAUL S. EPSTEIN..... 578
The Electric Field of a Magnetized Spheroid Rotating About the Axis of Magnetization. OSWALD ROGNLEY..... 609
Hall Effect and Specific Resistance of Silver Films. G. R. WAIT..... 615
Thermal Analysis at Low Temperatures. FRANK E. E. GERMANN..... 623
The Unilateral Dynamic Characteristics of Three-electrode Vacuum Tubes. JOHN G. FRAYNE..... 629
Index..... 665

Second Series *January, 1922* *Vol. XIX, No. 1*

THE

PHYSICAL REVIEW.

THEORY OF LONGITUDINAL VIBRATIONS OF VISCOUS RODS.

By W. G. Cady.

Synopsis.

Theory of Longitudinal Vibrations of Thin Rods, Taking into Account Damping.—Starting with the general differential equation of wave-motion in one dimension including a viscosity term, expressions are derived for the wave-velocity and logarithmic decrement in the case of *free vibrations* of the rod. The velocity is practically the same as for undamped vibrations, while the decrement is proportional to the viscosity and to the frequency. The equation is also solved for the case of *forced vibrations due to two simple harmonic forces at the ends*, equal in amplitude but opposite in phase. If the damping is small and frequency, $\omega/2\pi$, is near the resonance frequency, $\omega_0/2\pi$, the expression for the displacement of the end of the rod is very simple: $\xi = (4X_0 l/\pi G\delta)\cos\theta \sin(\omega t - \theta)$, where $\tan\theta = -2\pi(\omega_0 - \omega)/\omega_0\delta$, δ is the logarithmic decrement per period, l the length of the rod, X_0 the maximum value of the periodic stress, and G Young's modulus. It is shown that this expression may also be obtained by reducing the rod to an equivalent system possessing one degree of freedom.

§ 1. *General Equations of Wave-Motion in Rods.*—Our starting point is the following equation, which is exactly analogous to that for plane waves in an extended medium:[1]

$$\frac{\partial^2\xi}{\partial t^2} = P\frac{\partial^2\xi}{\partial x^2} + Q\frac{\partial^3\xi}{\partial x^2\partial t}. \qquad (1)$$

ξ is the displacement, at the time t, of that cross-section of the rod whose undisturbed coördinate is x. P is defined by the equation $P = G/\rho$, where G is Young's modulus and ρ the density. P is therefore the square of the wave-velocity in the absence of damping. As long as lateral effects can be ignored, it does not matter whether the material is isotropic or not: G is in any case the modulus along the axis of the rod. For brevity, we call Q the "viscosity," and treat it as a constant of the material, implying thereby that it is independent of the frequency. Its

[1] See, for example, Lamb's Dynamical Theory of Sound, 1910, Chap. VI.; or Lamb's "Hydrodynamics," 1916, Chap. XI.

possible dependence upon frequency can be tested by experiment. We shall leave aside all consideration of the mechanism by which the energy is dissipated in the material, and assume nothing further than that the frictional force is proportional to the rate of deformation. The dimensions of Q are $[L^2T^{-1}]$.

Since in this paragraph we are concerned only with damped sine-waves of any length, we may write the solution of (1) thus:

$$\xi = A\epsilon^{-jkx-bt}, \tag{2}$$

where

$$b = \eta + j\omega \tag{2a}$$

and A is a quantity dependent upon boundary conditions. k is the wave-length constant, η the attenuation-constant (that is, attenuation with time: attenuation in space need not be considered). $\omega = 2\pi f$ is the frequency-constant, or angular velocity.

On substituting (2) and (2a) in (1), and equating real and imaginary parts, we derive the following relations. For the attenuation-constant, we find

$$\eta = \frac{Qk^2}{2}. \tag{3}$$

The wave-velocity is

$$c = \frac{\omega}{k} = \pm\sqrt{P - \frac{Q^2k^2}{4}}. \tag{4}$$

On substituting this value in (2), we find for the displacement at any point

$$\xi = A\epsilon^{-Q^2k^2t/2}\cos(kx \pm \omega t). \tag{5}$$

From (4) and (5) it follows that

$$k = \frac{2\pi}{\lambda}, \tag{6}$$

where $\lambda = c/f$ represents the wave-length.

From (3), (5), and (6), the logarithmic decrement per period is found to be

$$\delta = \frac{\eta}{f} = \frac{Qk^2}{2f} = \frac{\pi\omega Q}{c^2}. \tag{7}$$

In order that a system of stationary waves may exist, the length l of the rod must be equal to an integral number of half wave-lengths. If this number is even, the center of the rod must also be free. Hence the wave-length constant for the fundamental or any harmonic is, by eq. (6), $k = \pi m/l$, in which m is any positive integer. The value for $m = 1$

gives the fundamental. From (4) and (7) it is evident that the damping increases, while the wave-velocity decreases, with increasing order of harmonics. For all practical purposes the change in velocity may be neglected; hence (always ignoring lateral effects) to a very close degree of approximation the harmonics have frequencies 2, 3, 4 $\cdots$ times the fundamental frequency.

§ 2. *Forced Vibrations.*—We will first solve the problem for the motion of a rod whose center is fixed, and at whose opposite ends two longitudinal simple-harmonic forces of like amplitude but opposing phases are applied; for the present paper originated in connection with a study of high-frequency vibrations of piezo-electric crystals and mathematically, the problem stated is identical with the problem of piezo-electric excitation, in which an impressed alternating electric field produces an alternating longitudinal mechanical stress, which at any instant is uniform throughout the rod, and numerically equal to the fictitious stress at either end. The identity of the two problems follows from the fact that the terminal conditions are the same, being expressed by eq. (11) below.

Assuming throughout that a steady state of vibration has been reached, so that the decrement η/f is compensated by an equal and opposite *increment*, we write the solution of (1) in the form

$$\xi = A\epsilon^{-j\omega t}, \tag{8}$$

in which A is a complex function of x, involving both the amplitude of ξ and the phase-difference between ξ and the impressed forces.

After the usual differentiations and substitutions, we find

$$\frac{\partial^2 A}{\partial x^2} = \gamma^2 A, \tag{9}$$

in which

$$\gamma^2 = \frac{\omega^2}{-P + j\omega Q}. \tag{10}$$

We take the origin at the center of the rod, so that the latter extends from $-l/2$ to $+l/2$. Let the impressed periodic stress at the ends of the rod have the form $X = X_0 \cos \omega t$, or, in exponential form, $X = X_0\epsilon^{-j\omega t}$ (dynes per cm.^{2}). Then at the end, where $x = l/2$,

$$G\frac{\partial \xi}{\partial x} = X_0\epsilon^{-j\omega t}, \tag{11}$$

where G is Young's modulus as before.

Equation (9) is now integrated, the constants being determined from (11) and the fact that, when $x = 0$, $\xi = 0$ and $A = 0$. In order to

save space, we omit the steps of the solution, which is most conveniently expressed in the form[1]

$$A = \frac{X_0}{G\gamma} \cdot \frac{\sinh \gamma x}{\cosh \frac{\gamma l}{2}} \cdot \tag{12}$$

Since, by eq. (10), γ is a function of the fundamental constants, it is evident that from (12) the amplitude and phase of the motion at any point along the rod can be derived.

From here on we shall be concerned only with the motion at the ends of the rod. The value assumed at the ends by A, which we will call A_0, is obtained by setting $x = l/2$ in eq. (12):

$$A_0 = \frac{X_0}{G\gamma} \tanh \frac{\gamma l}{2} \cdot \tag{13}$$

§ 3. In order to apply the last equation to actual cases, and in particular to use it for the determination of Q, it is necessary to reduce it to a more workable form. Upon reduction of (10), having regard to (3), (4), and (6), we find that, as long as Q is small,

$$\gamma = -\frac{\eta}{c} + jk = \frac{1}{c}(-\eta + j\omega). \tag{14}$$

This equation holds to a high degree of precision, even if the logarithmic decrement is as large as 0.1.

On substituting this value of γ in (13), and making obvious reductions and approximations, we arrive at the following expression for A_0, which is very accurate as long as Q is small enough to be ignored in (4) and n is small in comparison with ω:

$$A_0 = \frac{X_0}{Gk} \cdot \frac{4}{\frac{4\pi^2 n^2}{\omega_0^2} + \delta^2} \left(\frac{2\pi n}{\omega_0} + j\delta \right) = k_1 + jk_2, \tag{15}$$

in which the real coefficients k_1 and k_2 are written for brevity. ω_0 is the angular velocity at resonance (see footnote under § 4), and n denotes the difference $\omega_0 - \omega$, $\omega/2\pi$ being any frequency not far from resonance; n may therefore be regarded as a measure of the *dissonance* corresponding to any frequency.

From (15), together with (8), we readily find for the displacement at the end of the rod at any time t,

[1] In the abstract of this paper which appeared in the PHYSICAL REVIEW, Vol. 15, p. 146, 1920, the factor sinh γx was erroneously printed as sin γx.

$$\xi = \xi_0 \cos(\omega t - \varphi). \tag{16}$$

Here

$$\xi_0^2 = k_1^2 + k_2^2,$$

and

$$\tan \varphi = k_2/k_1 = \frac{\omega_0 \delta}{2\pi n}. \tag{17}$$

The phase-angle φ is 90° at resonance, and passes through most of the range from 0° to 180° in the region close to resonance.

§ 4. *Equivalent System with One Degree of Freedom.*—The fact that, with stationary waves, all parts of the rod agree in phase, suggests that a simple method of analysis may be reached by substituting for the actual rod an equivalent system possessing only one degree of freedom. We will therefore consider only the fundamental frequency.

The transition is easily made. In accordance with well-known principles[1] the equivalent mass M is found to be equal to half the actual mass of the rod, or $M = \frac{1}{2}\rho lbe$, these symbols representing density, length, breadth, and thickness respectively.

In place of Young's modulus we use the coefficient of stability, or "equivalent stiffness" g, which, close to the resonant frequency, and when the damping is small, is expressed as

$$g = M\omega_0^2 = \frac{\pi^2 beG}{2l}.$$

This equation follows from the resonance relation $\omega_0 = 2\pi f_0$, and the fact that $c = \sqrt{G/\rho} = 2f_0 l$. M and g correspond to L and $1/C$ in an electric circuit having concentrated, as contrasted with distributed, constants.[2]

The equation of motion is

$$M\frac{d^2x}{dt^2} + N\frac{dx}{dt} + gx = F_0 \cos \omega t. \tag{18}$$

The relation between x, the equivalent displacement, and the actual displacement ξ of the end of the rod, is given below. F_0 is the amplitude of the equivalent impressed force. The resistance factor, N, bears to the viscosity Q the relation $N = \pi^2 \rho beQ/2l$. This is proved by equating the decrement $N/2f_0M$ with the value given in eq. (7).

[1] See, for example, Lamb's Dynamical Theory of Sound, 1910, p. 13.

[2] Strictly, ω_0 is the angular velocity when the amplitude of the *velocity* dx/dt of the equivalent mass M is a maximum under forced vibrations. The maximum amplitude of equivalent *displacement* x comes (under forced vibrations) at the angular velocity $\sqrt{(g/M) - (N^2/2M^2)}$, while the angular velocity of free vibrations is $\sqrt{(g/M) - (N^2/4M^2)}$. The distinction between these three values may under ordinary circumstances be ignored.

The steady-state solution of (18) is

$$x = x_0 \sin(\omega t - \theta), \tag{19}$$

in which the maximum displacement is

$$x_0 = \frac{F_0}{\omega\sqrt{N^2 + \left(\omega M - \frac{g}{\omega}\right)^2}} = \frac{F_0}{\omega N}\cos\theta, \tag{20}$$

and in which

$$\tan\theta = \frac{\omega M - (g/\omega)}{N} = -\frac{2\pi n}{\omega_0 \delta} \tag{21}$$

approximately.

Equation (19) expresses the motion of the rod on what may be termed the "concentrated mass" method, in distinction from the "distributed mass" method first considered. It is easy to show that, near resonance, x and ξ agree in phase (cf. eqs. (17) and (21)). In order to make the amplitudes of x and ξ identical, the force F must be suitably expressed in terms of the impressed stress X_0. To this end, we consider the expression for ξ_0 in (17), and making use of eqs. (7) and (15) and the expressions for P, k, and N, we find that at resonance $\xi_0 = 2X_0be/\omega_0 N$. Upon equating this with the expression for maximum x_0 at resonance from eq. (20), namely $x_0 = F_0/\omega_0 N$, we see that F_0 must have the form

$$F_0 = 2X_0be. \tag{22}$$

This establishes the validity of the method of "concentrated mass," for all cases in which the damping and the range in frequency are both small enough for the expression for g to be satisfied to the desired degree of precision. Equations (19) to (21) are, under the restrictions just named, as accurate as those under the more general theory, and are much more convenient. Their application in the solution of problems with piezo-electric quartz rods, and in particular their use in determining the value of the coefficient of viscosity, must be reserved until later.

Finally, the following simple expression for the displacement of the end of the rod, in terms of the fundamental constants, may be derived from equations (19), (20) and (22), together with the expression for N in terms of Q:

$$\xi = \frac{4X_0 l}{\pi G \delta}\cos\theta \sin(\omega t - \theta).$$

It is easily verified that this equation also follows from equations (16) and (17) according to the more rigorous method of "distributed mass."

WESLEYAN UNIVERSITY,
June, 1921.

THE EFFECT OF FIELD DIRECTION ON MAGNETO-RESISTANCE.

BY C. W. HEAPS.

SYNOPSIS.

Variation of Magneto-resistance Effect with Direction of the Magnetic Field.—(1) *Theoretical discussion.* The fact that in general the resistance of a conductor differs slightly according to whether the magnetic field is transverse or longitudinal, may be due to two factors. First, there may be a direct effect of the field on the motion of the electrons, as predicted by the electron theories of J. J. Thomson and others. But the author shows that if Townsend's method of developing the electron theory of conduction is adopted, the variation of resistance comes out zero. Second, the effect of the magnetic field may vary with the arrangement of crystals in the conductor, and therefore with the position of the conductor if the crystal structure is anisotropic. (2) *Experiments with cast bismuth, pressed graphite and rolled cadmium* are described in which a small bar or sheet of the material was placed in each of three mutually perpendicular positions in a field of 7,000 or 8,000 gausses, and resistance measurements were made for various positions of the rotatable Weiss magnet. For bismuth and cadmium the variations found are not symmetrical around the current direction and are evidently due chiefly to the crystal structure. For graphite the variations when the field was rotated in the plane of the sheet were within the experimental error, therefore both factors were inappreciable. These results are not conclusive but they suggest that the Townsend theory is correct. If so this magneto-resistance effect may be completely explained by assuming a change in the number and mean free period of the conducting electrons which depends not only on the magnetic field but on its direction with reference to the crystal axes.

WHEN a metallic conductor of electricity is placed in a magnetic field there may be for any given conductor two factors which affect the magnitude of the resulting change of resistance. These two factors are the crystalline structure of the specimen and the angle between the magnetic field and the electric current. Experiments dealing with the latter of these two factors have concerned themselves chiefly with the two cases where the magnetic field is either transverse or longitudinal with respect to the electric current. Lenard [1] found that a longitudinal field produced a smaller resistance increase in a bismuth wire than a transverse field. The writer[2] has obtained a similar result in the case of tellurium, bismuth, lead sulphide, cadmium, zinc, gold, and graphite. Patterson[3] states that his experiments on copper appear

[1] Ann. d. Phys., 39, p. 619, 1890.

[2] PHYS. REV., 10, p. 366, 1917; Phil. Mag., 24, p. 813, 1912.

[3] Phil. Mag., 3, p. 643, 1902.

to indicate a slightly smaller effect for the longitudinal field. Barlow,[1] on the other hand, studying a plate of bismuth, found very little difference in the resistance change for the two directions of the field. The curves given by Barlow seem to indicate that a longitudinal field produces the greater increase of resistance.

The experiments cited above were performed for the most part with specimens of undetermined crystal structure, hence it is conceivable that the crystal structure of the specimens differed for the different directions of the magnetic field. In such a case we cannot say definitely that it is the orientation of the current with respect to the magnetic field which affects the magnitude of the resistance change; it might well be the lack of isotropy of the specimen which produces this effect.

The effect of crystal structure on magneto-resistance has been investigated for graphite by Roberts.[2] Roberts concludes that the resistance increase of this substance is independent of the direction of the electric current with respect to the field, depending only on the angle between the crystal axis and the magnetic field. De Haas[3] experimented with antimony and proved that for this metal the orientation of the crystal axis is of considerable importance. His experiments are not decisive in the matter but he draws the conclusion that the angle between the directions of the magnetic field and of the current is of no importance, at least to a first approximation.

Now if the conclusion of de Haas regarding this question is correct it is a matter of considerable importance in its bearing on the electron theory of metallic conduction. De Haas, considering the free electron theory, says that the influence of the magnetic field on the free paths of the electrons must be considered as negligible, and that such theories as that of J. J. Thomson[4] which try to calculate the phenomenon from the direct effect of the field on the free electrons cannot possibly give the right result. Now it is generally assumed in the literature of the subject that the electron theory of metals does afford an explanation of magneto-resistance, at least for transverse fields and non-ferromagnetic metals. A number of writers[5] following the general method of Sir J. J.

[1] Ann. d. Phys., 12, p. 921, 1903.
[2] Ann. d. Phys., 40, p. 467, 1913.
[3] Konink. Akad. Wetensch. Amsterdam, 16, p. 1110, 1914.
[4] Rapports presentes au Congres International de Physique, III., p. 138, 1900.
[5] E. P. Adams, PHYS. REV., 24, p. 428, 1907.
Gans, Ann. d. Phys., 20, p. 293, 1906.
Livens, Phil. Mag., 30, p. 526, 1915.
Heaps, PHYS. REV., 10, p. 366, 1917.
Richardson, Electron Theory of Matter, p. 439.
Righi, "I Fenomeni Elettro-atomici," p. 401.

Thomson have derived expressions for the increase of resistance of a metal when placed in a transverse magnetic field. These expressions involve the relation

$$\frac{dR}{R} = C\left(\frac{e}{mv}\right)^2 \frac{l^2}{v^2} H^2, \tag{1}$$

where dR is the increase produced in the resistance R by the magnetic field H, e/m is the ratio of the charge to the mass of the electron, l is the mean free path of the electron, v is its velocity of agitation, and C is a constant depending on the type of theory adopted. Properly speaking, this expression for dR/R refers only to the increase of resistance produced when a longitudinal field is rotated into the transverse position and when the specimen is isotropic. It cannot be supposed, therefore, that calculations of mean free paths, etc., by the use of this formula will be of much value when experimental results are obtained from specimens of unknown crystalline structure or when the dR/R of the above formula is taken to represent the entire effect of a transverse field as was the case in Patterson's calculations.

If the conclusions of de Haas are correct, *i.e.*, that there is no intrinsic difference between the effects of a transverse and a longitudinal field, then we should expect the theoretical expression for dR/R of equation (1) to come out equal to zero. As a matter of fact, one form of the theory gives this result, as may be shown in the following manner.

Let us adopt the ordinary assumptions of the free electron theory of metals, assuming collisions between electrons and molecules to be like those between hard elastic spheres. With these conditions J. S. Townsend has developed an expression for the velocity with which a group of free electrons drifts through an aggregation of molecules under the combined influence of electric and magnetic fields.[1] In addition to assuming elastic collisions Townsend neglects persistence of velocities, assumes that the free periods of electrons may vary from zero to infinity, and lets the number of free periods, out of a total number N, comprised in the time interval between t and $t + dt$ be equal to $(N/T)e^{-t/T}dt$, where T is the mean free period.

If the magnetic field H acts along the z axis and the electric fields X and Y act along the x and y axes, respectively, Townsend's theory gives for the drift velocities U and V along the respective x and y axes:

$$U = \frac{e}{m} T \frac{(X - Y\omega T)}{1 + \omega^2 T^2}, \tag{2}$$

$$V = \frac{e}{m} T \frac{(Y + X\omega T)}{1 + \omega^2 T^2}, \tag{3}$$

[1] Electricity in Gases, p. 100.

where $\omega = He/m$. In Townsend's development Y is assumed zero. It is to be noted that the deduction of the above formulæ does not involve approximations like those made in the references cited above, where other methods are used.

If we consider a plate of metal lying in the x, y plane and carrying a current along the x direction the condition that there shall be no current along the y direction requires V to be zero. There must, therefore, exist an electric field given by $Y = -X\omega T$ in the metal to prevent the electron current along y. It is the potential difference due to this field which is observed as the Hall effect.

If we substitute this value of Y in equation (2) we get

$$U = X\frac{e}{m}T. \tag{4}$$

Hence the drift velocity along x under these conditions is not affected by H so long as T remains unchanged. The current density along x is given by

$$I = neU = \frac{ne^2}{m}XT, \tag{5}$$

where n is the number of free electrons per unit volume. Thus if n, T, and X are not functions of the magnetic field, the current I is independent of H and we have

$$\frac{dR}{R} = 0. \tag{6}$$

We may now consider a form of conductor in which $Y = 0$. Let the current enter at the center of a flat circular plate and leave it at its periphery, and let the plane of the plate be normal to H. This is the arrangement used by Corbino.[1] Under these conditions the magnetic field causes a circular current to flow in the plate, and since there is no banking up of electrons to produce the Hall e.m.f. we may assume that Y, taken as perpendicular to a radius, is zero. In this case the radial current may be represented by

$$I_r = neU = \frac{ne^2}{m}X\frac{T}{1+\omega^2T^2}. \tag{7}$$

Hence

$$\frac{I_0 - I_r}{I_r} = \frac{dR}{R} = \omega^2T^2, \tag{8}$$

where I_0 is the current when H is zero. Thus with the Corbino arrangement the specific resistance of the metal will be found greater than in

[1] Phys. Zeits., 12, pp. 561, 842, 1911.

the case where the Hall e.m.f. is allowed to develop. Under the ordinary conditions of measuring magneto-resistance, however, the Hall e.m.f. is always present, so it is not permissible to assume $Y = 0$ in developing the theory, as has been done by many previous writers.

It thus appears according to Townsend's theory that there should ordinarily be no change of resistance produced in a metal by the direct action of the magnetic field on the motion of the electrons between collisions. Townsend's theory is therefore in agreement with the conclusions of de Haas.

The theory of magneto-resistance has been worked out by Gans and by Livens.[1] These writers follow the method of H. A. Lorentz in assuming a slight departure from Maxwell's law of distribution of velocities among the electrons. The effect of the Hall e.m.f. is not neglected. The formula obtained by Gans for small magnetic fields is

$$\frac{dR}{R} = \frac{12 - 3\pi}{8}\left(\frac{e}{m}\right)^2 \frac{l^2}{v^2} H^2. \qquad (9)$$

This formula assumes collisions to be like those between elastic spheres and here also only the difference between the effect of a transverse and a longitudinal field is contemplated. Thus the theory of Gans does not agree with the conclusions of de Haas. Livens assumes molecules to act as centers of force and gets a more general formula. He points out that if the potential energy of an electron repelled by a molecule at a distance r is given by $m/2 \cdot (\mu/r)^s$ then when $s = 4$ the value of dR/R should be zero.[2] It appears therefore that if de Haas is correct in his conclusions regarding experiment—*i.e.*, that the difference between the longitudinal and transverse magneto-resistance effects is zero—then the theory of Livens demands a special type of field around a molecule. If on the other hand we adopt Townsend's method of handling the problem then the molecules and electrons may act like solid elastic spheres and we still get agreement with the conclusions of de Haas.

It seemed to the writer that further experiments were necessary in order to establish a theory, hence the effect of the direction of the magnetic field on the resistance of bismuth, graphite, and cadmium has been investigated. The magnetic field was furnished by a large Weiss electromagnet capable of being rotated about a vertical axis. The angular position of the magnet could be read from a scale at the base of the instrument. The pole-pieces were 10 cm. in diameter and were set so that the faces were 3.03 cm. apart. The specimen to be examined was

[1] L.c.

[2] The formula of Livens reduces to that of Gans if $s = \infty$.

supported between the pole-pieces so that the electromagnet could be rotated without disturbing it. This specimen was inserted in one arm of a Wheatstone bridge, balancing being accomplished by a shunt arrangement somewhat after the fashion described by Crandall.[1] A Leeds and Northrup moving coil galvanometer of resistance 13 ohms and sensitivity 11 mm. per microvolt was used. Since the current through the electromagnet had a value of 8 or 10 amperes, and since no special precautions were taken to maintain constancy of temperature, the resistance of the specimen was found to change very slowly during the time required to obtain a set of observations. A correction for this temperature effect was made by periodically repeating a standard resistance measurement of the specimen and assuming a linear resistance change during the interim between these periodic measurements. As a matter of fact when a set of observations was taken with reasonable rapidity no very great error was introduced by temperature effects. The bridge current was made as small as was convenient—of the order of 0.01 ampere—and was kept flowing during the whole time of taking a set of observations.

The process of making observations consisted in balancing the bridge with the specimen between the poles of the magnet but with no exciting current flowing. The magnetizing current was then set up and a new balance of the bridge obtained. With this magnetizing current kept constant the bridge was balanced for different angular positions of the electromagnet and values of $\delta R/R$—which is here the total increase of resistance of the specimen divided by the resistance in zero field—were calculated for these various positions of the magnet. In calculating $\delta R/R$ the resistance of the copper wires leading to the specimen was carefully allowed for, though it was unnecessary to consider the effect of the field on the resistance of these leads since copper shows very small magneto-resistance effects. When the specimen was removed from between the poles it was found that revolving the magnet did not affect the bridge balance. Spurious effects, such as the influence of the magnet on the zero position of the galvanometer can thus be considered as negligible.

The bismuth used in this investigation was rated by Merck as about 98 per cent. pure. A specimen was made by cutting a thin bar of rectangular cross-section from a thin plate of cast bismuth. The dimensions of this bar were roughly $1.1 \times 0.15 \times 0.05$ cm., the shortest dimension being perpendicular to the plane of the original large plate of metal. Experimental results for this bar are given in Fig. 1, where the

[1] PHYS. REV., 2, p. 343, 1913.

lenght of the radius vector from o to any point of a curve is set equal to $\delta R/R$ and the angular position of this radius vector is determined by the orientation of the magnet. Results are plotted only for positions of the magnet comprised between 0° and 180°, as duplication results in the case of larger angles. The field strength was 7,000 gausses and the temperature was that of the room—about 23° C.

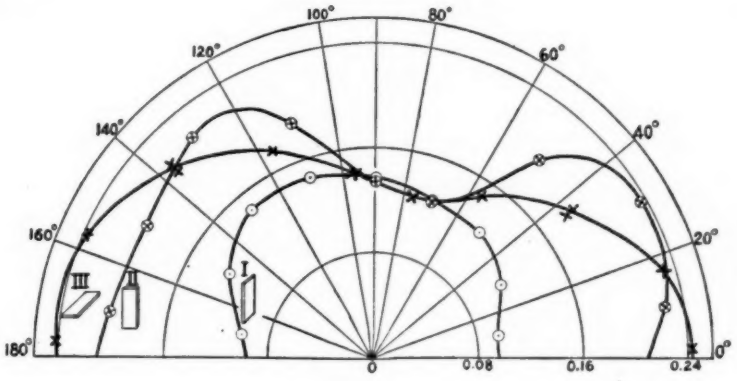

Fig. 1.

Three curves were obtained for this specimen, corresponding to three different settings of the bar between the poles of the magnet. The three-dimensional diagrams of the bar placed beside the respective curves will make the arrangement clear. The magnetic field is always in a horizontal plane, the plane of the figure. In curve I. the length and thickness of the bar are in this horizontal plane, the breadth in a vertical plane. The magnetic field for this curve was thus always perpendicular to the breadth of the specimen. In curve II. the breadth and length of the bar were in a horizontal plane; the magnetic field was thus always perpendicular to the shortest dimension of the specimen. In curve III. the breadth and thickness of the specimen were in a horizontal plane; the magnetic field was here, therefore, always perpendicular to the length of the specimen and to the current. It will facilitate interpretation of the curves to note that the relative orientation of field and specimen may be determined for any point on the curve by imagining the diagram of the specimen to be shifted without rotation to the desired point of the curve.

The lack of symmetry of these curves is an indication of a complex crystalline structure in the bar. Curves I. and II. coincide, within the limits of error of the experiment, at the angle 90°. This coincidence is, of course, to be expected, for at this point the relative arrangement of specimen and field is identical for the two curves. Curve III. should

coincide with curve II. at 0° and 180° for the same reason. Similarly curve III. should have the same value at 90° as curve I. at 0° and 180°. The fact that curve III. does not have these values ($\delta R/R$ being too great in each case) is very probably due to a slight error in measuring lead resistances, for in setting the specimen for III. after I. and II. had been obtained it was necessary to disconnect the old leads and put on new ones. The specimen itself had a very small resistance—about 0.05 ohm—while the resistance of the leads was 0.0442 ohm. The Wheatstone bridge, though sensitive to changes of resistance, was not very accurate for the measurement of absolute values of small resistances. Slight inaccuracies in measuring these small resistances would affect the calculated value of $\delta R/R$ but would be of no consequence in making comparisons along any one curve. Another cause of error might lie in the altered resistance of the soldered joint between lead wire and specimen. No measurement of this junction resistance could be easily made. A third cause of error lies in the difficulty of setting the specimens accurately in position.

Certain general conclusions may be drawn from the curves of Fig. 1. In curve I. the effect of a longitudinal field (measured by the length of the radius vector at 90°) is greater than the effect of a transverse field (measured by the length of the radius vector at 0°). For curve II. the converse is true. It appears obvious that crystalline structure plays a very important rôle in the phenomenon of magneto-resistance. Since curves I. and II. are not symmetrical with respect to the 90° position (the direction of current flow) we must conclude that crystalline structure is producing a distortion of the curves. It thus appears impossible in this bismuth specimen to separate the effect due to current direction—if there is such an effect—from the effect due to crystal structure. We may conclude, however, that crystalline structure is a very important factor because curve III. obtained with the current always transverse shows great variations as the field changes direction. The conflicting results of Barlow and Lenard cited above can now be explained as arising solely from the different arrangement of crystalline axes with respect to the field. The complicated nature of the results obtained with bismuth might have been foreseen from the work of E. van Everdingen[1] who found that when a bismuth crystal is placed in a magnetic field of arbitrary direction its resistance may be represented by an ellipsoid with three unequal axes. In general, then, a cast bismuth plate is not apt to possess a plane in which a magnetic field may alter its direction without encountering dissimilar crystalline conditions.

[1] Konink. Akad. Wetensch. Amsterdam, III., p. 407, 1901.

Some further experiments were performed on wire made by forcing molten bismuth into a vertical capillary tube. It might be expected that for all directions of a magnetic field perpendicular to the wire the same magneto-resistance effect would be noted. Such, however, proved not to be the case, a small but unmistakable dissymmetry being observed. When the specimen was placed horizontally and the magnet rotated from the transverse to the longitudinal position the latter position was found to give the smaller effect. The greatest increase of resistance was observed when the magnetic field made an angle of 72° with the length of the wire.

For the experiments on graphite the specimen was made from the ordinary powdered graphite, consisting of small crystalline particles, which is used for lubricating purposes. This powder was compressed by means of a hydraulic press into the form of a thin plate on the top of an ebonite block. The graphite was made to adhere to the ebonite by a thin coat of beeswax and resin which had been previously applied to the ebonite. Two brass screws had been set into the block with their heads flush with its surface, so that by cutting away part of the graphite plate with a razor blade a thin bar was obtained with its ends resting in close contact with the screw heads. Copper wires were soldered to the ends of the screws and used for connecting the specimen into the Wheatstone bridge. The dimensions of the bar were roughly 1.2 × 0.15 × 0.03 cm. and its resistance was 1.15 ohms at 26.5° C. This resistance increased in one week to 1.23 ohms. Probably this increase was due to gradual readjustment of strains in the specimen.

The curves of Fig. 2 represent results obtained with this graphite in a

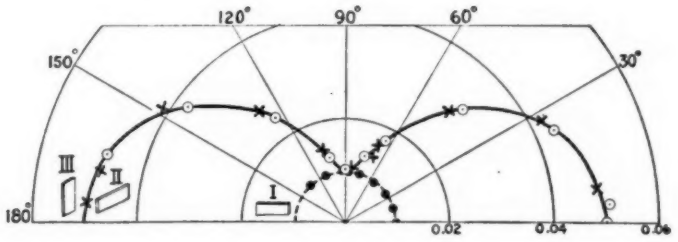

Fig. 2.

field of 8,000 gausses. The orientation of the specimen for any particular set of observations is represented diagrammatically as in the previous case with bismuth. It appears from curve I. which is practically a semicircle, that as long as the magnetic field is in the plane of the specimen we get the same change of resistance irrespective of the direction of the

field. There is a slight deviation of the points from the curve. It is so slight, however, as not to show up in the graph and is probably caused by inaccurate setting of the specimen. The points for curve II. are marked with crosses, those for curve III. with circles. These two curves coincide to a fair degree of accuracy. The obvious interpretation of the curves is as follows.

In the plane of the specimen the arrangement of the small crystals is such that any direction in this plane is the same as any other direction as far as the average crystalline structure is concerned. If this is the case then the circular nature of curve I. implies that the direction of the electric current is of no importance as regards the magnitude of the resistance change produced by the magnetic field. Curves II. and III. indicate that the average crystalline structure in a direction perpendicular to the plate is different from that parallel to the plane of the plate, but the fact that these two curves coincide is again evidence that the change of resistance is independent of the relative directions of field and current.

The conclusions which Roberts[1] reached in his study of large graphite crystals are corroborated by the above experiments on a conglomerate. In the writer's previous work with graphite powder[1] it was concluded that $\delta R/R$ for a transverse field was greater than that for a longitudinal field by 8×10^{-4}. Differences of that order of magnitude were obtained in the various observations of curve I., but it seems probable that these differences were the result of experimental errors arising from temperature changes or inaccurate adjustment of the specimen. The longitudinal field of curve I. gave a value of $\delta R/R$ greater by 1.5×10^{-4} than the $\delta R/R$ of the transverse field. It is difficult to adjust such small specimens accurately with respect to the magnetic field and a small deviation from accuracy can produce quite large changes in $\delta R/R$. Possibly the writer's previous results can be explained as due to inaccurate adjustments.

The experiments on cadmium were performed on a specimen made as follows. A disk of the metal was sawed from a round bar. This disk was put through a rolling mill a number of times, and always in the same direction so that finally a long thin sheet of cadmium was secured. A section of this sheet was then cemented with wax flat upon a glass plate and cut with a razor into the form of a grid 0.9 cm. wide and 2.5 cm. long, containing 16 strips of cadmium connected in series and having a total resistance of 1.71 ohms. This grid was supported between the poles of the magnet in a horizontal position and examined for magneto-resistance in a field of 7,600 gausses. A fairly large bridge current had to be used with this metal in order to get sufficient sensitivity, so that troublesome

[1] L.c.

temperature effects were introduced. However, by taking in rapid succession a number of observations for the transverse and longitudinal positions of the magnet it was found that $\delta R/R$ for the transverse field was greater by 2.1×10^{-5} than for the longitudinal field. The specimen was then placed so that the strips of the grid were vertical, the field being always transverse to the current. A series of observations was taken with the field alternately perpendicular and parallel to the plane of the grid. It was found that $\delta R/R$ was greater by 5.2×10^{-5} for the perpendicular field than for the field parallel with the surface of the grid. Since the current was here always transverse to the field it is to be concluded that differences of crystalline structure are responsible for the difference of 5.2×10^{-5}. As regards the difference of 2.1×10^{-5} this might be due either to differences of current direction or to the anisotropic character of the strips.

In previous work by the writer on cadmium strips made by hammering no difference was detected in $\delta R/R$ for a transverse field whether normal or parallel with the surface of the strips, while a longitudinal field was found to produce a smaller effect than the transverse field. If this specimen made by hammering was really isotropic as regards magnetic effects then it appears that the current direction is a factor in determining the resistance change of cadmium. It cannot be said with certainty, however, that such an isotropic character was produced by this method. On the other hand, rolling the specimen does introduce an anisotropic character. Since the drawing of metal into the form of wires is more analogous to the rolling method than to the hammering method we might expect that cadmium wires would exhibit magneto-resistance effects for a longitudinal field which are different from those for a transverse field solely because of the anisotropic nature of the wires. We cannot say, therefore, that experiments which have been performed on cadmium have either proved or disproved the theory of magneto-resistance outlined above. Tests of other metals such as zinc, gold, copper, etc., could not be made because of insufficient sensitivity of the apparatus.

The general conclusions to be drawn from the above experiments are as follows. Influences of crystal structure are so great in the metals bismuth and cadmium that any effect arising from the direction of the current cannot be differentiated from effects arising from crystal structure. In the case of graphite, if the direction of the electric current with respect to the magnetic field is of any importance at all in changing the magnitude of magneto-resistance effects then any such resulting change is very small compared with the effect of crystal structure on magneto-

resistance. The experiments with graphite may be considered as supporting the correctness of equation (6). If equations such as (9) hold true it must be supposed that l/v is very small, at least in graphite. On the whole, one must conclude that calculations such as those of Patterson, in which electron concentrations and mean free periods of electrons in different metals are determined from measurements of magneto-resistance, are by no means to be relied upon. The effect of crystal structure is too vital a factor to neglect. Furthermore the theory as developed above by Townsend's method indicates that there should ordinarily be no effect of a magnetic field on resistance (provided this effect is limited to direct action on the electrons' motion) and it appears that there is some experimental evidence in support of this theory. If we adopt the Corbino arrangement, however, and use equation (8) it is possible to calculate the mean free period of the electrons in the metal. Essentially this method has been used by the writer in a previous paper,[1] and it is perhaps surprising that the results obtained agree as well as they do with the results calculated by Patterson from J. J. Thomson's equation.

Suppose now that we consider equation (6) as essentially correct, that is, suppose a magnetic field does not alter the resistance of a metal by virtue of any direct action on the motion of an electron between collisions. Then in order to explain the change of resistance observed experimentally we shall have to consider the factors of equation (5) and determine which are functions of the magnetic field. It is conceivable that the electric field X in the region where the electrons move may be modified by the action of the magnetic field on the molecules of the metal. If polarization electrons play any part in modifying the internal field [2] it is not impossible that a magnetic field, say by changing the orientation of molecular magnets, should affect this field produced by the polarization electrons. However, since there appears to be no relationship between the direction of the magnetic field with respect to X and the resistance increase which this field produces in the metal, it seems safe to say that X is not dependent on H. The average distance between molecules is probably changed by a magnetic field [3] so that the free period T will depend on H. It has been suggested by a number of writers[4] that n should be affected by a magnetic field. If we imagine in the metal a system of electrons moving in open and closed orbits—

[1] PHYS. REV., 12, p. 340, 1918.

[2] See Richardson, Phil. Mag., 23, p. 614, 1912.

[3] See E. P. Adams, l.c.

[4] E. van Everdingen, Konink. Akad. Wetensch. Amsterdam, III., p. 177, 1900.
La Rosa, N. Cim., 18, p. 39, 1919.
Heaps, l.c.

as suggested by Richardson[1]—it is not difficult to understand why a magnetic field might alter equilibrium conditions in such a system and thus change n.

If we let n_0, T_0, and I_0 be the respective values of n, T, and I when the magnetic field is zero then

$$I_0 = n_0 \frac{e^2}{m} X T_0. \tag{10}$$

Combining this equation with (5) gives

$$\frac{I_0 - I}{I} = \frac{\delta R}{R} = \frac{n_0 T_0}{nT} - 1. \tag{11}$$

This equation, in the light of Townsend's theory, is probably the most general expression for magneto-resistance. The factor given by equation (1) does not appear. It is to be noted that both n and T may depend upon the direction of the field in a metallic crystal as well as upon the magnitude of the field.

The experimental work described in this paper serves to emphasize the importance of crystal structure in relation to magneto-resistance. The question might well be asked as to whether crystalline structure is not essential to the production of the phenomenon. Becker and Curtiss[2] have found that bismuth films made by cathode sputtering, and therefore presumably amorphous, show no magneto-resistance effects until after they have been subjected to a heating process which crystallizes the metal. As far as the writer is aware, a magnetic field has never been found to change the resistance of a strictly non-crystalline substance,[3] so it is not impossible that an orderly arrangement of the molecules is essential for the manifestation of the phenomenon. The settling of this question, however, requires more extensive experimental data than is available at present.

THE RICE INSTITUTE,
HOUSTON, TEXAS.

[1] Electron Theory of Matter, p. 463.

[2] PHYS. REV., 15, p. 457, 1920. See also Richtmyer and Curtiss, PHYS. REV., 15, p. 467, 1920.

[3] Patterson found an effect with liquid mercury but later work (see Zahn, Jahrbuch der Radioaktivität und Elektronik, 5, p. 197, 1908) indicates that the observed increase of resistance is to be ascribed to electrodynamic actions.

ON THE PRINCIPLE OF COMBINATION AND STOKES' LAW IN THE X-RAY SERIES.

By D. Coster.

Synopsis.

New X-Ray Line γ_6 for Elements Tantalum to Uranium.—The two emission lines γ_2 and γ_3 are found to be in reality three lines. The new line γ_6 has *wave-lengths* of 1110, 1072, 842, 630 and 593 $\times 10^{-11}$ cm. for Ta, W, Tl, Th and U respectively. In the cases of Ir, Pt and Au it is too near γ_2 and in the cases of Pb and Bi too near γ_3 to be separated.

New Absorption Lines of the M-series.—Ma_1, Ma_2 and Ma_3 have been measured for *bismuth*, and Ma_4 and Ma_5 for *thorium and uranium*, the *wave-lengths* being, respectively, 4762, 4569, 3894, 2571, 2388, 2385 and 2228 $\times 10^{-11}$ cm.

Identification of the Line for Tantalum and Tungsten.—For all the elements Os to U, the difference in wave-lengths of $\beta_2 - \beta_5$ is equal to about 29 $\times 10^{-11}$ cm. If this difference is the same for Ta and W also, the β_5 lines for these elements must be the lines 1220.8 and 1250.6, respectively, instead of those ordinarily assumed.

Measurement of La_1 and La_2 for Tungsten by a different method gave the same wave-lengths as those found by Duane and Patterson.

Relationships between Emission and Absorption X-Ray Spectra.—(1) *Stokes's law* seemed not to hold in the case of W. It was thought the apparent discrepancies might be due to an error in the measurement of La_1 and La_2, but these measurements were found correct; however, the discrepancies disappear if we identify β_5 with 1220.8 instead of 1212.5 and if we associate γ_2 and γ_3 with La_2 and the new line γ_6 with La_2. (2) *Relations between frequencies.* The frequencies of seven of the L-series lines are each approximately equal to the difference between the frequency of an La-line and that of an Ma-line.

IN a previous note I called attention to the fact that Stokes' law holds for the L-series of the elements Pt – U. For W however the limiting wave-lengths La_1 and La_2 seemed to be much larger than the wave-lengths β_5 and γ_2 respectively; and I suggested a possible source of error in the method by which Duane and Patterson determined the absorption-wave-lengths of tungsten. On account of the great theoretical importance of this matter I have repeated the measurements, but with essentially the same result as these authors. To avoid every kind of complication I proceeded in the following manner. Use was made of a water-cooled bulb with aluminium-cathode, constructed by Prof. Siegbahn and described by A. Hadding.[1] This bulb has the great advantage

[1] Zeitschrift für Physik, *3*, 369, 1920.

that the slit of the spectrograph can be placed very near to the anticathode. It was driven with a current up to 15 *mA* and a tension of about 40 *kV*. First the absorption spectrum was taken with an iron anticathode and the absorbing substance placed between the bulb and the slit. Part of the slit was covered with a thick lead screen, so that only the upper part of the plate was exposed. In order to eliminate effects due to inhomogeneities of the absorbing screen, this was turned round during the exposure. Then the iron anticathode was replaced by a tungsten one, the absorbing substance was removed and by a suitable adjustment of the lead screen an emission-spectrum could be obtained on the lower part of the same plate. The plates demonstrated quite obviously that the absorption wave-lengths are considerably larger than the wave-lengths thus far denoted by β_5 and γ_2. For purposes of control the La_2 and the Lγ-lines of gold and lead were taken in the same way; but, as might be expected, no such a difference was found here.

Now we may escape from the theoretical difficulty lying in an exceptional condition of the tungsten-spectrum by assuming, as already has been done by Hoyt[1] and by Dauvillier,[2] that the β_5-line of this element has not as yet been identified accurately. As may be seen from Table I.

TABLE I.[3]

	U.	Th.	Bi.	Pb.	Tl.	Au.	Pt.	Ir.	Os.	W.	Ta.
$\beta_2 - \beta_5$ ..	28.58	28.49	30.63	30.48	29.56	29.55	29.40	29.87	28.38	29.41	30.32

the wave-length difference $\beta_2 - \beta_5$ is nearly constant from U down to Os. If now we take for β_5 of W the line 1212.8 and for Ta the line 1250.6 then for these elements the difference $\beta_2 - \beta_5$ likewise remains nearly the same.

As regards the line γ_2, the situation is more complicated than could be anticipated formerly. Recently I have taken new photographs of the emission-lines of the elements Ta – U with the same metal bulb.[4] I find, that instead of the lines thus far denoted by γ_2 and γ_3, we must assume three lines.[5] One of these, which we may call γ_6, lies at a constant wave-length-distance from γ_1. The frequency-difference $\gamma_{\gamma 6} - \gamma_{\beta 5}$ is

[1] Proc. Nat. Ac. Sc. Washington, Nov., 1920.

[2] Comptes Rendus, 11 April, 1921.

[3] The wave-lengths are given in cm. $\times 10^{-11}$.

[4] See Zeitschrift für Physik, *6*, 158, 1921.

[5] An analogous hypothesis was pronounced some time ago in a letter from Mr. Wentzel to the author.

the same as the difference $\gamma_{\beta 1} - \gamma_{\alpha 2}$. For the elements Ta and W, γ_6 is of very small intensity relatively to the other γ-lines; but, in the neighborhood of Pt there is a sudden rise in its intensity. A similar change has been observed for β_5. Moreover, there are two lines γ_2 and γ_3, which lie at about the same wave-length-distance from one another and which both have about the same intensity. Only for the elements Ta, W, Tl and U was γ_6 separated from the lines γ_2 and γ_3, but where complete separation did not occur, a corresponding broadening and darkening of γ_2 or of γ_3 could usually be observed.

TABLE II.

	γ_1	γ_6	γ_2	γ_3	$\gamma_1 - \gamma_6$	$\gamma_2 - \gamma_3$
U	612.83	592.6	604.4	597.0	20.23	7.4
Th	651.03	630.1			20.93	
Bi	810.65	787.4	792.9	787.4	23.25	5.5
Pb	837.08	813.70	818.2	813.70	23.38	4.5
Tl	865.29	841.7	844.7	837.9	23.59	6.8
Au	924.37	901.25	901.25	859.68	23.12	5.57
Pt	955.45	931.7	931.7	925.6	23.75	6.1
Ir	988.41	963.6	963.6	956.6	24.8	7.0
W	1,095.53	1,072.0	1,065.84	1,059.65	23.53	6.19
Ta	1,134.71	1,110.0	1,102.0	1,096.2	24.71	5.8

In Table II. the wave-lengths of the lines γ_1, γ_6, γ_2 and γ_3 and the wave-length-differences $\gamma_1 - \gamma_6$ and $\gamma_2 - \gamma_3$ are given in cm. 10^{-11}. It seems to be most probable that only γ_6 is connected with the La_2-discontinuity, whereas γ_2 and γ_3 both belong to La_3.

Thus far only three absorption-discontinuities Ma_1, Ma_2 and Ma_3 in the M-series of U and Th were known, these corresponding to the measurements of Stenström. I have succeeded in measuring two other discontinuities Ma_4 and Ma_5 of shorter wave-length of U and Th and Ma_1, Ma_2 and Ma_3 of Bi, with the following provisional results:

TABLE III.

	Ma_1	Ma_2	Ma_3	Ma_4	Ma_5
U	3,491	3,326	2,873	2,385	2,228
Th	3,721	3,552	3,058	2,571	2,388
Bi	4,762	4,569	3,894		

The values for Ma_1, Ma_2 and Ma_3 of U and Th were taken from Stenström's dissertation. As is seen from Table IV. the frequencies of the lines l, η, α_1, α_2, β_1, β_3 and β_4 are each equal to the difference of an L-absorption-frequency and a M-absorption-frequency.

TABLE IV.[1]

	1	$L\alpha_1$–$M\alpha_1$	η	$L\alpha_2$–$M\alpha_1$	α_1	$L\alpha_1$–$M\alpha_1$	α_2	$L\alpha_1$–$M\alpha_2$
U.....	855.84	855.1	1,134.95	1,131.8	1,003.23	1,002.17	990.37	989.21
Th.....	819.19	818.8			955.78	954.80	944.08	943.15
Bi.....					789.54	797.43	790.20	789.35

	β_1	$L\alpha_2$–$M\alpha_2$	β_3	$L\alpha_2$–$M\alpha_3$	β_4	$L\alpha_2$–$M\alpha_4$
U.....	1,269.08	1,265.87	1,286.29	1,285.78	1,222.53	1,220.86
Th.....	1,194.94	1,193.15	1,211.67	1,209.77	1,155.00	1,151.86
Bi.....	959.93	958.16	973.85	975.86		

Apart from the accidental errors there seems to be a systematic deviation between the line-frequency and the difference of the absorption-frequencies, so that we may conclude that there must be a small systematical error either in my measurements of the lines or in the work of Duane and Patterson. If this is the case, it is most probable that the same error should be found in the measurements of all the elements. We should therefore be inclined to conclude that for the elements Pt, Au, Tl, Pb and Bi, and perhaps for W also, the absorption-wave-lengths are a little shorter than the corresponding wave-lengths β_5 and γ_6 and that the difference should correspond to a difference of about 5–10 volts.

A superficial examination of my plates does not give the impression that the absorption-wave-lengths are shorter than the corresponding emission-wave-lengths. But the selective absorption commences with a faint white line, which may be tentatively considered as an image of the slit just as the black emission-lines are. This shows that the electron "prefers" the absorption of a very definite wave-length, which is at the same time the smallest frequency it is able to absorb. To estimate this frequency we are to measure the middle of the white absorption-line relatively to the middle of the black emission-lines. I hope to have later the opportunity to study this problem with the required accuracy with the photometer-method.

It gives me great pleasure to express my thanks to Professor Siegbahn, who put the apparatus of his laboratory at my disposal and advanced the work by his great interest.

LUND, SWEDEN, PHYS. LAB. OF THE UNIVERSITY,
June, 1921.

[1] The frequencies have been calculated in multiples of the Rydberg constant (limiting frequency of the Lyman series of hydrogen).

THE MOBILITY OF ELECTRONS IN PURE NITROGEN.

BY LEONARD B. LOEB.

SYNOPSIS.

Mobility of electrons in pure nitrogen was determined, using the Rutherford alternating potential method, for pressures ranging from *75 to 600 mm.*, for frequencies ranging from *7,000 to 150,000 cycles* (obtained from an audion oscillating circuit), and with electric fields of from 10 to 100 volts/cm. Reduced to atmospheric pressure the mobility found is of the order of 10,000 cm. sec./many times the highest value previously obtained, and it was observed to vary with pressure and electric field according to the equation $K = 571{,}000/(21 + 760\ V/pd)$, where V/d is in volts/cm. and p in mm. of Hg. A discussion of the possible sources of error shows that none can be responsible for this variation with V/pd, and the form of the mobility curves confirms this variation. A *theoretical interpretation* of these results on the basis of the Townsend equation for electron mobility leads to the conclusions: that if the Townsend theory is correct (1) either the *energy lost at each impact* of an electron with a nitrogen molecule or the *mean free path* must be a function of V/pd, and (2) in either case the mean free path for velocities of the normal agitation must be about 22 times the mean free path of the gas molecules instead of $4\sqrt{2}$ times, as given by the kinetic theory.

I. INTRODUCTION AND PREVIOUS RESULTS.

THE results obtained in a study of the attachment of electrons to neutral molecules to form negative ions,[1] indicated that the mobility constant for the electrons in a gas at atmospheric pressures was apparently not constant. It thus became of interest to determine the mobility of electrons in gases in which they move unattached to molecules at atmospheric pressures in order to discover what conditions govern the variations of mobility. The mobilities of electrons in pure argon and nitrogen at atmospheric pressures were first determined by Franck,[2] who obtained 206 cm./sec. for the highest value in the former and 145 cm./sec. for the highest value in the latter gas. Haines[3] measured the mobilities of electrons in nitrogen and found mobilities as high as 506 cm./sec. while the average value obtained lay at about 367 cm./sec. In hydrogen he found the highest value to be 170 cm./sec. Finally

[1] Loeb, L. B., PHYS. REV., Vol. XVII., No. 2, pp. 106–107, Feb., 1921.

[2] Franck, J., Verh. d. Deut., Phys. Ges., 12, p. 613, 1913.

[3] Haines, Phil. Mag., 6, 30, p. 503, 1915.

Wellisch[1] working with what he thought were purely electronic carriers in CO_2 obtained mobilities which varied from 256 cm./sec. to much lower values. In H_2 he got a value of 1,700 cm./sec. for the mobility.

II. Experimental Method.

In the present work the mobilities were determined on the photo-electrons liberated from a brass plate (P), Fig. 1, struck at grazing incidence by ultraviolet light from a quartz mercury arc, using the Rutherford alternating current method. The experimental method of making the determinations was essentially the same as that used by the writer in an earlier[1] paper on negative ions. The gas used was pure nitrogen in which it has been shown that the electrons do not attach to form[2] ions.[5] The method of purification was carried out as described in the paper[5] showing this to be the case. A reduction of the contamination of the gas while in the measuring chamber by vapors arising from sealing waxes and organic materials used in closing the chamber was accomplished by the use of a new ionization chamber. It was similar to the one previously described, except that the rubber gasket was replaced by a ground steel joint with a mercury seal.

The mobilities were measured by the use of the high-frequency oscillations from two vacuum tube oscillators of the Western Electric "E" tube type operating in parallel. The circuits used are shown in the diagram, Fig. 1. In some cases the oscillations were taken directly off the primary condenser, and led to the plate P of the measuring system. In many others they were taken from the condenser of a secondary circuit closely coupled to the primary circuit, and tuned to resonance with it. The alternating potential was varied in the measurements by changing the resistance (r) in the plate circuit of the oscillators. This change in the resistance resulted in a gradual change in the frequency with the voltage, sometimes causing a reduction in frequency as the voltage decreased and at other times causing an increase, depending on the constants of the oscillating circuit used at the time. It therefore became necessary to measure the frequency of the oscillations at several values of the voltage in order to obtain accurate values of the mobility. This was done with a wave meter. The frequencies thus obtained appeared to be in good accord with those computed from the circuit constants, where computation was possible. The alternating potentials were measured with Kelvin type static voltmeters which were calibrated from time to time against a standard Weston voltmeter. A large number

[1] Wellisch, E. M., Am. Jour. Science, July, 1915.

[2] Loeb, L. B., Proc. Nat. Acad. Sciences, 6, No. 7, pp. 335–441, July, 1920.

of mobility curves were taken covering frequencies ranging from 30,000 cycles up to 150,000 cycles per second, and pressures of nitrogen from 600 mm. to 75 mm. of Hg. It was found possible to study the mobility curves with potentials which varied from 20 volts to 140 in all cases, and from 30 volts to 300 in a few cases. Below 20 volts it was not satisfactory to make mobility measurements as the oscillations became rather unsteady, though some determinations were carried down to less than ten volts. The plate distances PE used in the measuring chamber were 2 cm. and 1.5 cm.

III. RESULTS.

In Figs. 2 and 3 are given some of the typical mobility curves, obtained by plotting the electrometer current against the electrostatic voltmeter reading under the conditions of frequency and pressures indicated in the legend. The intercepts of these curves with the voltage axis (or in the case of the curves with markedly asymptotic feet the intercepts of the extrapolated straight portions of the curves with the axis) yield the value of V_0 from which the mobility constant of the electron K is determined by the equation,

Electron Mobility Curves, Current Plotted against Voltage.

No.	Press. in Mm. p.	Frequency n.	Intercept Corrected V_0	Mobility u.	Mobility Constant K.
1.....	610	28,000?	7.6	34,800	28,000
2.....	597	31,800	14.3	21,000	16,500
3.....	607	49,200	24.8	17,350	13,850
4.....	599	53,100	28.5	17,600	13,700
5.....	615	62,200	32.6	17,900	14,600
6.....	606	68,000	40.1	14,850	11,850
7.....	604	76,000	57.3	11,600	9,230
8.....	599	75,000	66.8	10,650	8,400
9.....	625	95,500	80.1	11,100	9,200
10.....	603	82,200	90.7	8,600	6,800
11.....	597	88,000	94.0	8,200	6,440

Electron Mobility Curves, Current Plotted against Voltage.

No.	Press. in Mm. p.	Frequency n.	Intercept Corrected V_0	Mobility u.	Mobility Constant K.
1.....	302	82,000	38.7	18,000	7,340
2.....	303	98,000	74.4	11,500	4,610
3.....	166	93,000	32.4	25,100	5,500
4.....	166	115,000	104.3	9,650	2,100
5.....	75	141,000	75.4	16,250	1,610
6.....	74.5	118,000	48.6	21,200	2,090
7.....	75.0	91,000	21.0	38,000	3,750

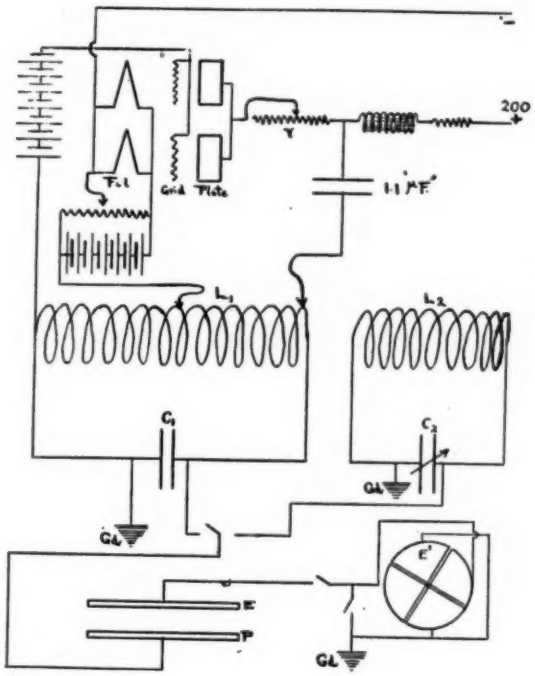

Fig. 1.

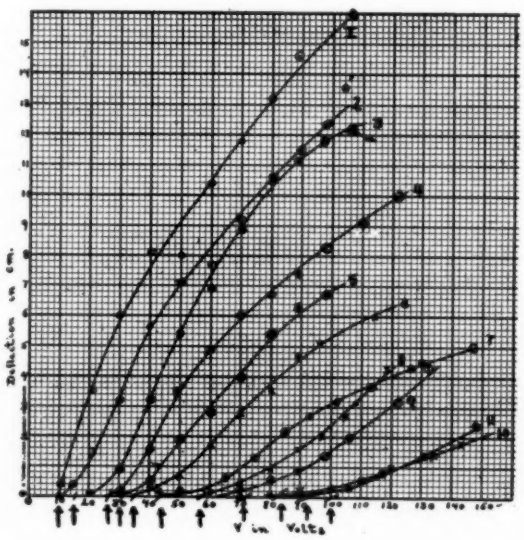

Fig. 2.

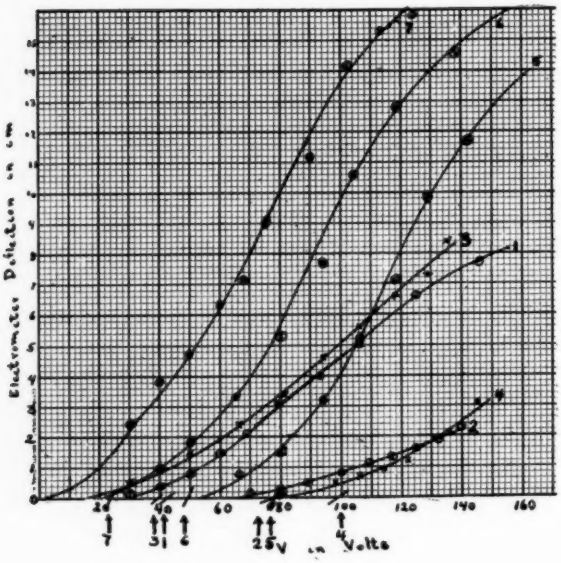

Fig. 3.

$$K = \frac{\pi n d^2 p}{\sqrt{2} V_0 760},$$

where n is the frequency of the oscillation, d is the distance between the plates PE, and p is the pressure in mm. of Hg. In this case as before the value of V_0 as obtained from the intercept must be corrected by a deduction of 4.6 per cent. of its value in the case where $d = 2$ cm. to take care of the drop in potential across the electrometer while the readings were being made. A discussion of the sources of error and the nature of the curves obtained will be postponed until a later point in the paper.

The mobility constants thus obtained were of the order of magnitude of 10,000 cm./sec. volt/cm. This is a value far in excess of any obtained by the previous investigators. It seems likely that the successful measurement of such high values was due to three essential differences between the writer's mode of procedure and those of the previous observers. All three of the workers mentioned above used electrons generated by the action of alpha particles on the gas in an auxiliary chamber (*i.e.*, they used the Franck modification of the Rutherford method), while the writer used photoelectrons freshly liberated from one of the plates. Furthermore the writer used extreme measures not only for purifying his nitrogen gas but also for reducing the contamination from the chamber to a minimum. That this was successful was shown by the fact that

nitrogen kept in the chamber for as long as 10 days still yielded electron mobilities which were of the same value as when the gas was freshly prepared. Finally the use of very high frequencies by the writer enabled only the purely electronic carriers to be studied. The other workers had used frequencies at most as high as 3,000 cycles per second, while the writer never used frequencies below 7,000 cycles and generally worked above 20,000 cycles. The mobility constants were furthermore found to vary through wide limits. It was found that by plotting K the mobility constant obtained from the equation above for a given value of d and p as a function of V_0, the voltage intercept, a smooth curve could in general be drawn through the points. For each value of d or p a

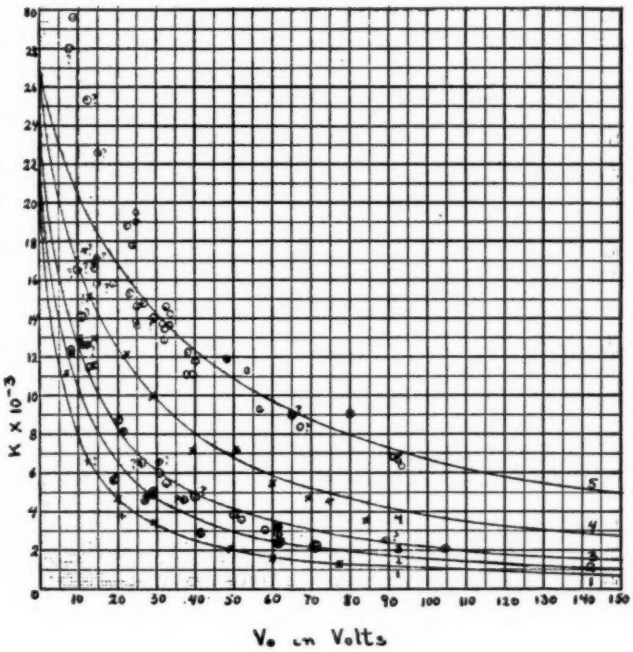

Fig. 4.

Mobility Constant K Plotted against V_0 Corrected.

No.	Press. p in Mm.	Plate Distance d in Cm.
1	75.5	2.00
2	160.0	2.00
3	304.0	2.00
4	150.0	1.52
5	606.0	2.00

different curve was obtained. The curves obtained for all the determinations are shown in Fig. 4. The legend gives the conditions under

which they were obtained. The curves were hyperbola of the form $K = a/(b + V)$. Furthermore if the equation for one curve with a given d' and p' be obtained (*i.e.*, that for curve (1) at $d = 2$ cm. and $p = 75$ mm. which is $K = 113/(4.18 + V)$) then it will be found that the equation for any of the other curves may be obtained from it, by multiplying the constants of this equation by $dp/d'p'$. Where d' and p' are the values of pressure and distance corresponding to the curve whose equation is known, and d and p are the pressure and distance corresponding to the curve to be determined. In other words, all the results may be expressed by means of the equation

$$K = \frac{571{,}000}{21 + (760V)/pd},$$

within the limits of error of the measurements. That this is the case is seen from the fit of the observed points to the curves plotted from the above formula. In noting this agreement it is well to point out that too much reliance must not be placed on the values of K obtained for intercepts at values of V_0 less than 10 volts. At this point owing to the extrapolation necessary to get the intercepts of the mobility curves, the values of K can easily vary as much as 30 per cent. or more. It is also interesting to note that as the pressures increase the deviations of the points on both sides of the theoretical curves become greater. This is in part due to the fact that a variation of say 10 per cent. from the curve at the higher values of the mobility will actually be greater in magnitude, than in the case where the mobilities are low.

The result obtained seems quite surprising and it is natural to inquire into the effects of possible sources of error before accepting them.

IV. Sources of Error.

(*a*) *Effect of Wave Form.*—The first question which arises is as to whether changes in the wave form of the oscillations due to higher harmonics might not influence the results. The equation given for the mobility for the case of an oscillation of pure sine-wave form was $K = 2.22nd^2/V_0$. For a square wave form, the equation becomes $K = 2nd^2/V_0$. The error introduced by a change from a sine-wave to a square wave, an extreme case, would be to give values of the mobility which are 11 per cent. too high. Errors due to neglect of wave form will therefore in general be slight.

(*b*) *How Does the Variation of K with V Affect the Mobility Curves and Hence these Determinations?*—One may next inquire as to the effect of the variation of the mobility constant K with the voltage on the

shape of the mobility curves. A theoretical study of the shape of the mobility curve for photoelectrons, liberated while the alternating field is acting, assuming a square wave form and that K is a constant yields the equation $I/I_0 = (V - V_0)/V$; where I is the current, I_0 is the maximum possible current, and V_0 is the critical value of the voltage when the electrons can just get to E. It has been found that for the case of the negative *ions*, where K is constant, such an equation reproduces the form of the observed mobility curves with a square wave fairly well. For the case of a sine-wave the equation for the electronic current as a function of the voltage for K constant is found to be $I/I_0 = 1/\pi \cos^{-1}\{(2V_0/V) - 1\}$. The two types of curves for two different values of V_0 are shown in Fig. 5.

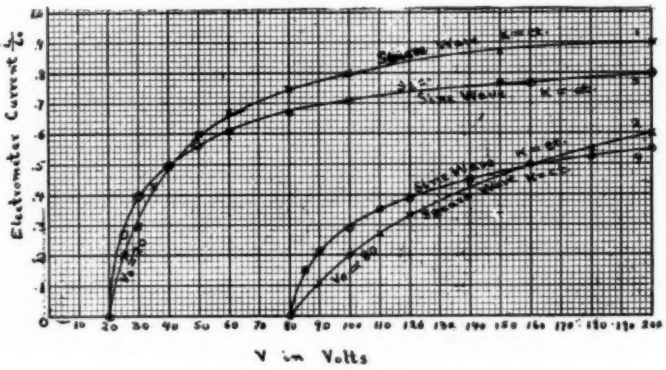

Fig. 5.

Theoretically computed curves for variation of current with an alternating voltage for ions or electrons for which the mobility constant K is constant. This is computed for photoelectrons.

No.	Wave Form.	V_0.
1	square	20.0
2	square	80.0
3	sine	20.0
4	sine	80.0

It is seen that they are similar in general form except that the curve for the sine-wave rises a bit more abruptly at first. If now K varies, as was found experimentally to be the case, viz., $K = a/(b + V)$, the equation for the square wave becomes one of the form,

$$I/I_0 = \left(1 - \frac{V_0 + V/b}{V}\right),$$

where $V_0 = nd^2/K_0$ and $K_0 = a/b$. On plotting this equation we get the types of curves seen in Fig. 6 for the positive values of the current.

The value of the intercept will if substituted in the mobility equation for a square wave form yield (*e.g.*, $K = 2nd^2/V_0$) values of K that vary as K was assumed to vary. For the case of the sine-wave with $K = a/(b + V)$ it is impossible to determine the equation directly. The mobility curves may, however, be determined for a given set of conditions, by means of graphic integration. This was done, and the curves so obtained for three values of V_0 are shown in Fig. 6. These curves are not unlike those obtained for the square wave form and differ from them chiefly in the sharper rise of the curve near V_0 and a slightly lower value of

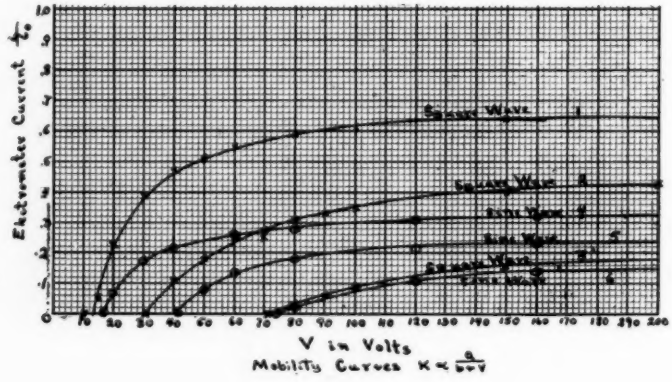

Fig. 6.

Theoretical curves for variation of current with an alternating voltage for electrons, for which K is a variable with V according to the equation, $K = a/(b + V)$. Values of the constants taken from experiment. Curves 1, 2 and 3 correspond to the curves for a square wave under the conditions of the experiment at frequencies of 30,000, 50,000 and 70,000 cycles and 606 mm. pressure. Curves 4, 5 and 6 correspond to the curves with a sine wave form and the same frequencies respectively at a pressure of 760 mm. The plate distance here is 2 cm.

i/i_0. It is at once seen that the apparent saturation values of the curves of Fig. 6 do not nearly rise to the values attained with K a constant (see Fig. 5). It is also to be noticed that these saturation values are progressively lower the greater the value of V_0. This peculiarity which had been observed in all the experimental curves (see Figs. 2 and 3), had puzzled the writer for some time, until it was found to be a natural consequence of the variation of K with voltage.

(*c*) *Discussion of the Asymptotic Feet of the Mobility Curves.*—There is another peculiarity of the curves which up to the present has not been satisfactorily explained. Since the electrons in crossing the plates make millions of collisions before they cross, it is to be expected that they all move with a nearly uniform velocity of drift in the direction of the field. Thus the intercepts of the mobility curves with the voltage axis should be quite sharp. Actually this is not the case. The mobility curves

have asymptotic feet which become more pronounced as the value of V_0 for the curves increases and so drags out these portions of the curves over a greater range of voltages. A study of these feet indicate that they are due to a factor which increases the electrometer current more and more rapidly as the quantity $V_0 - V$ decreases to 0. This portion of the curves is also apparently increased in importance by a decrease in pressure or a decrease in the distance between the plates. At present the only plausible explanation of these feet is that they are due to the formation of negative ions in the gas. Now the gas used, though quite pure, is not absolutely so. Therefore while the electrons are oscillating back and forth in the space between the plates some of them attach to form negative ions. These, owing to their low mobilities, remain between the plates. In the course of the million cycles which elapse while a measurement is being made this process might build up such a gradient of ions that some of these will diffuse to the upper plate E and register a current. The gradient built up will be the sharper (and hence the number diffusing will be the greater), the nearer the electrons get to crossing the distance between the plates, *i.e.*, the smaller $(V_0 - V)$, and the smaller the distance d. As a decrease in the pressure increases the coefficient of diffusion of the ions, it would also increase the number that get across. An attempt was made to remove these feet by putting a weak but constant field in the circuit in such a manner as to remove the accumulated ions as they were formed. The reduction in prominence of the feet produced by a field of .75 volt per cm. was however not sufficiently great to permit the explanation offered to account for the *whole* effect. It does however account for a portion of it.

(*d*) *A Possible Effect of the Unsymmetry of the Waves Relative to the Ground Potential.*—Finally one must consider the possibility that a state of affairs exists in the oscillating system such that the oscillations, that is the positive and negative crests of the waves, are unsymmetrically placed with respect to the ground potential. This condition, though unlikely in this type of circuit is, however, important enough when present to merit a brief discussion. Consideration will show that when the negative side of the wave has a higher peak value than the positive side, by even a small fraction, it will become impossible to obtain any mobility curves at all. For the electrons will be accelerated by the field for a longer time interval and by a greater average force when the phase is negative than when it is positive. Accordingly the electrons advance towards the electrometer plate E a distance d', and are pulled back from it a distance equal to d'', where $d' > d''$. Thus since the measurements last over a million or more cycles the electrons

get across the plates for very small potential differences. With the positive peaks higher than the negative peaks, the ions will cross as in the case for symmetrical waves. However, the value of V_0 taken from the curves, as registered on the static voltmeter, will be higher than the actual value of the driving potential. Also the time over which the potential acts will be shorter than that assumed from the frequency. The mobility as computed from readings under these conditions, will be lower than the actual value. If such an unsymmetry should actually occur and increase systematically with the increase in frequency, it might produce results analogous to those obtained. Calculation shows that to produce a mobility which is but 78 per cent. of the true value it requires that the height of the negative peak be .82 that of the positive peak. If the mobility were to be reduced to one fifth its true value the negative peak would have to be .33 as high as the positive peak. With a Braun tube oscillograph no unsymmetry of the oscillations over a considerable range of frequencies could be detected, within the limits of resolution of the oscillograph used. The instrument could easily have shown unsymmetry of .82 between the two peaks. Other tests also failed to reveal any unsymmetry. In no case was unsymmetry with the negative side greater than the positive side manifested by any mobility measurements. Finally the diversity of coils and the diversity of connections used in obtaining the oscillations should a priori have caused no systematic change of any unsymmetry present such as would be required to yield the results described.

V. The Interpretation of the Results on the Basis of Townsend's Electron Mobility Equation.

(*a*) *Modification of the Langevin Equation by Townsend.*—An interpretation of the variation of the mobility found is difficult. Townsend[6] suggests that the Langevin[7] equation for ionic mobilities may be applied to the case of electrons. In doing this one must assume that owing to the elastic impacts which the electron makes with the more massive molecules its energy of agitation in the gas (*i.e.*, its temperature), increases as it moves through the gas, due to the action of the field. Townsend[8] gives the Langevin equation, when it has been reduced to the case of a body of mass small compared with that of the molecules as

$$K = .815 \frac{e}{m} \frac{\lambda}{c}, \tag{1}$$

where e is the electronic charge, m its mass, λ its mean free path, and

[6] Townsend, J. S., "Electricity in Gases," Oxford, 1914, p. 174 ff; Phil. Mag., Vol. 40, 1920.

[7] Langevin, P., Ann. de Chimie et de Phys., 8, 5, p. 245, 1905.

[8] Townsend, J. S., Electricity in Gases, p. 180.

$\bar{c}$ its mean velocity of thermal agitation. This equation must be modified according to Townsend by multiplying the $\bar{c}$ by a quantity $\sqrt{k}$ before it may be applied to the case of the electron in an electric field. (A similar equation is given by Lenard[12] except that in his equation the constant term is .5 instead of .815 and that he uses α_1 in place of $\sqrt{k}$.) This quantity $\sqrt{k}$ must then be a quantity of such a form that when it is multiplied by c it gives the average velocity of agitation of the electron in the field. If the maximum energy which the electron can acquire due to the field alone be called W, the average energy of the electron as a result of the heat motions and the field must be given by $[2(m\bar{c}^2/2) + W]/2$. The average velocity of the electron as a result of thermal agitation and the action of the field will then become $\sqrt{\bar{c}^2 + W/m}$, and hence the quantity $\sqrt{k} = \sqrt{1 + W/m\bar{c}^2}$.

(*b*) *Consequences of This Equation for Elastic Impacts.*—Now the value of W will depend on how the electron gains and loses energy in the field. If, as is the case for the most part in helium and the monatomic gases, the impacts are nearly perfectly elastic, it will according to the quantum theory continue to gain in energy of agitation until it acquires enough energy to suffer an inelastic resonance impact. At the high pressures under which the experiments above were conducted the electrons probably exceeded their resonance potentials R very little before suffering a resonance impact. Also if the total fall of potential between the plates was several times that of the resonance potential, the value of W was probably close to $R \times e$, where e is the electronic charge. Thus the average velocity of the electrons in this case would approximate a value of the form $\bar{c}\sqrt{1 + Re/m\bar{c}^2}$. The mobility of the electron on Townsend's assumptions would then be given by

$$K = .815 \frac{e}{m} \frac{\lambda}{\bar{c}\sqrt{1 + \frac{Re}{m\bar{c}^2}}},$$

Since $Re/m\bar{c}^2 >> 1$,

$$K = .815 \frac{e}{m} \frac{\lambda}{\bar{c}\sqrt{\frac{Re}{m\bar{c}^2}}}.$$

It is seen at once that this mobility would be a constant depending solely on R, and would not vary as a function of p or d as was experimentally found to be the case.

(*c*) *Consequences of this Equation for the Case of Partially Elastic Impacts.*—Thus one must conclude that the electron in nitrogen does not

behave as in helium, but that it loses a fraction of its energy (f) at each impact. If this is the case Compton[12] has shown that the value of W for the electrons, after a sufficient number of impacts, approaches a terminal value given by

$$W = Ue = .575 \frac{Ve\lambda}{d\sqrt{f}},$$

which is in agreement with a result arrived at by Pidduck in a paper entitled "The Abnormal Energy of an Electron in a Gas." [11] Here V is the potential across the plates, d is the distance between them, and f is the fraction of the energy of the electron lost at each impact. The value of $\sqrt{k}$ then becomes

$$\sqrt{k} = \sqrt{1 + .575 \frac{Ve\lambda}{\sqrt{f} dmc^2}},$$

and hence

$$K = .815 \frac{e}{m} \frac{\lambda}{c\sqrt{1 + .575 \frac{Ve\lambda}{mdc^2\sqrt{f}}}}$$

on the basis of Townsend's reasoning. This of course only holds for the values of a terminal energy of agitation which is less than the resonance value Re discussed above. That such a fractional loss of energy (f) actually may occur in nitrogen gas is shown both by the results of Franck and Hertz,[9] and those of Compton.[10] The mobility of the electron in the equation above varies with the voltage, the distance between the plates, and with the pressure; for λ the mean free path varies with the pressure. The form of the variation of K with these quantities for the equation would however be

$$K = \frac{a}{\sqrt{1 + \frac{bV}{pd}}}$$

while the observed equation makes

$$K = \frac{a}{b + \frac{V}{pd}}.$$

The two equations, observed and theoretical, may only be caused to agree in case either $\sqrt{f}$, or λ, vary as a function of (V/pd). In the former case agreement can be had only if $\sqrt{f}$ increases as the value of V/pd decreases. That is when the loss of energy of the electron on impact

[9] Franck and Hertz, Verh. d. Deut. Phys. Ges., 15, p. 34, 1913; Verh. d. Deut. Phys. Ges., 15, p. 373, 1913.

[10] Compton, K. T., PHYS. REV., Vol. XI., 3, p. 196, March, 1918.

[11] Pidduck, F. B., Proc. Roy. Soc., A 88, p. 300, 1913.

[12] Lenard, P., Ann. der Phys., Vol. 40, p. 409, 1913.

decreases as the energy of the electron increases. It is well known that such a variation of the loss of energy of the electron occurs on impacts when the electrons move with high speeds. Whether such can be the case at the low values of the velocities of the electrons dealt with here is, however, doubtful.

(*d*) *Value of the Mean Free Path of Electrons in Nitrogen Deduced from the Results on the Basis of Townsend's Theory.*—Finally if in the general experimental equation for the mobility of the electron,

$$K = \frac{571{,}000}{21 + \frac{760V}{pd}},$$

the value of V be set equal to 0, it is seen that K becomes 27,200 cm./sec. volt/cm. On the Townsend point of view this represents the value of the mobility when the electron has a velocity equal to its velocity of thermal agitation in equilibrium with the gas molecules. The value of this mobility may be computed from the Townsend equation when V approaches 0, by setting the mean free path of the electrons as $4\sqrt{2}$ that of the nitrogen molecules. The value thus obtained is 6940 cm./sec. volt/cm. This marked difference in the two values may be interpreted as meaning one of two things. Either that one is not justified in applying the Langevin equation to the case of the electron at all, a point of view which may after all be correct; or that the mean free path of the electron is 3.9 times as great as that which has been assumed above.[13] If the latter view point is correct it is likely that the definition of the "mean free path" of an electron in a gas must be distinctly modified.

In conclusion the writer wishes to express his thanks to Prof. R. A. Millikan for his kind advice and criticism. The writer also desires to acknowledge his indebtedness to Dr. A. J. Dempster for many valuable suggestions and criticisms related to the more theoretical aspects of this paper.

NATIONAL RESEARCH FELLOWSHIP,
RYERSON PHYSICAL LABORATORY,
UNIVERSITY OF CHICAGO,
June 30, 1921.

NOTE ADDED DECEMBER 14, 1921.

In applying these measurements to electrons in H_2 gas a new oscillating system was used in which the voltage V could be varied from 10–200 volts without changing the frequency. Measurements made on N_2 checked the values here given within the limits of experimental variation.—R. B. LOEB.

[13] This high value of the mean free path may have some justification in view of the recent results of Mayer and Ramsauer on argon at velocities of electrons below 3 volts. Ann. der Phys., Vol. 64, p. 451 and p. 613, 1921.

AN EXPERIMENT ON ELECTROMAGNETIC INDUCTION AND RELATIVE MOTION.[1]

BY W. F. G. SWANN.

SYNOPSIS.

Electromagnetic Induction; Charging Effect of the Rotation of an Iron Sphere in a Uniform Magnetic Field.—The electromagnetic theory of this effect among others was developed by the author in a former paper. The answer which the theory gives is perfectly definite, but the nature of the problem is such that one who is accustomed to think in terms of moving lines of force may be uncertain as to the proper basis of attack and may even, quite plausibly, arrive at a result one third as great as it should be. Therefore *an experimental verification of the theoretical formula* seemed worth while carrying out. It was found that when an iron sphere with a radius of 3.17 cm. was rotated at rates ranging from 3,250 to 6,430 r.p.m., in a magnetic field ranging from 3,930 to 6,070 gauss, the changes of potential of an insulated copper shell surrounding the iron sphere, which were observed when the magnetic field was reversed ranged from 0.0114 to 0.0359 volt, while the corresponding calculated values ranged from 0.0116 to 0.0353 volt. In view of the experimental difficulties the fact that the average difference between observed and calculated changes of potentials for the seventeen sets of readings is only 1.7 per cent. is considered a satisfactory verification of the theory.

INTRODUCTION.

IN a recent paper[2] the present author has discussed the so-called "Moving-line Theory" of electromagnetic induction, and has described the sense in which it must be interpreted in order to give the equivalent of the Maxwellian electromagnetic theory. It was shown, moreover, that, apart from effects of the order of magnitude of the ratio of the square of the velocity of a particle of matter to the square of the velocity of light, and concerned with the alteration of the constitution of the matter as a result of the motion, the electromagnetic theory contains a complete answer in the case of any problem concerned with the motion of magnets or of bodies in magnetic fields. There is no ambiguity such as frequently appears to present itself in discussions on moving lines of magnetic force, and concerned with whether the æther or the lines of magnetic force do or do not move with the bodies associated with them.

It appeared that if a symmetrical magnetized body were rotated about its axis of magnetization, the motion of the magnetized material would

[1] Presented at the meeting of the American Physical Society, December 28–30, 1920.

[2] PHYS. REV., Vol. 15, pp. 365–398, 1920.

give rise to an electric polarization perpendicular to the axis of rotation, in the case of a non-conductor; but, in the case of a conductor, the state of polarization would bring about a volume and surface distribution of charge which would completely annul the electric field at all external and internal points. At least this would be so in the absence of the "Motional Intensity" resulting from the motion of the substance of the conductor in its own magnetic field. This motional intensity results in the superposition of an electrostatic distribution upon the condition described above, and this electrostatic distribution is the sole representative of the ultimate effect as regards electric field. Moreover, not only is the electrostatic distribution produced by the motional intensity the sole representative of the ultimate electrical effect in the case of the rotation of a permanent magnet, but it is also the sole representative of this effect in the case of the rotation of a magnetizable body, or of course of a non-magnetizable body in an external field. In all cases the motional intensity is given by $[\mathbf{v}\cdot\mathbf{B}]/c$, where $\mathbf{v}$ is the velocity of the element of matter, c the velocity of light, and $\mathbf{B}$ the magnetic induction as measured by the force on a fixed unit magnetic pole placed in a thin drum-shaped cavity supposed cut out of the body, with its flat ends perpendicular to the flux.

In general, the electrostatic distribution resulting from the motional intensity results in a potential other than zero at the axis of rotation if this axis is insulated; and, in this case, moreover, the total charge associated with the electrostatic distribution is zero. If the axis of rotation is earthed, an additional charge comes to the body to bring the axis to zero potential, and it is of course possible to measure this charge.

In the paper above referred to, the present author has worked out the case for a sphere rotating in a uniform magnetic field, and has shown that, in the case where the direction of rotation is clockwise as viewed by an observer looking in the direction of the axis of magnetization, the potential produced at the axis of rotation when insulated is $-2\pi Bna^2/3c$ e.s.u., where B is the induction in e.m.u., n the frequency of rotation, a the radius of the sphere, and c the velocity of light. Under this condition, the total charge on the rotating sphere is zero, so that if this sphere were surrounded by an insulated sphere, the potential of the latter sphere would not be altered by stopping the rotation or reducing the magnetic field to zero; for, the various elements of charge on the inner sphere, which elements all add up to zero, would produce contributions to the potential of the outer sphere depending only upon their amounts, and independent of their positions. If we should now earth the axis, the inner sphere

would take a charge, and the potential of the outer sphere would alter by an amount V which we shall proceed to calculate. Our problem amounts to considering the simple electrostatic problem where the inner sphere is devoid of rotation and is initially at zero potential, and finding the potential V to which the outer sphere and its attachments rise when, without altering the total charge on them, we charge the inner sphere to potential $2\pi Bna^2/3c$; for, a superposition of this latter solution upon that corresponding to the case where the axis was insulated, the inner sphere was in rotation, and the outer sphere was at zero potential, obviously satisfies all the conditions of the problem.

If subscript unity refers to the inner sphere, and subscript two to the outer sphere and any attachments such as an electrometer, and if q_{22} and q_{12} refer to the usual coefficients of capacity and induction, we have, since the charge on the outer sphere and its attachments is zero as a whole:

$$q_{22}V + \frac{2\pi Bna^2}{3c} q_{12} = 0.$$

Hence[1]

$$V = -\frac{2\pi Bna^2}{3c}\frac{q_{12}}{q_{22}}. \qquad (1)$$

The object of the present paper is to verify this result experimentally, and to this extent to verify the conclusions upon which it is based. To any one who takes his stand firmly on the basis of the Maxwellian theory, it would appear, in accordance with the discussion in the paper above referred to, that no reason exists for suspecting that (1) might be other than true. Presumably nobody would doubt its validity in the case of a non-magnetic sphere, one of copper for example, where there would be no temptation to raise considerations as to whether any part of the field took part in the rotation, and where the simple results of the motional intensity represent all that is to be anticpiated on any view of the matter. In the case of an iron sphere rotating in the field, however, it is quite conceivable that one who thought entirely in terms of moving lines might have some doubts as to the result, depending upon the manner in which he regarded the phenomenon.

[1] *Correction to previous paper.* The last step in the derivation of formula (23) is not quite correct, although the formula itself is correct. In the case of two concentric spheres of radii a and b, b being the larger, it is easy to show that $q_{12} = -ab/(b-a)$, and $q_{22} = b^2/(b-a)$. Thus, $q_{12}/q_{22} = -a/b$, which shows formula (23) of the previous paper to be in agreement with (1) above. In the case where the outer sphere is connected to an electrometer, b must be replaced by the measured capacity of the sphere and electrometer as stated in the paper, but one of the a's must also be replaced by $ab/(b-a)$, so that the resulting effect is to increase in the ratio $b/(b-a)$ the quantities 5×10^{-4} volt and 3×10^{-2} volt calculated on page 394.

Thus, for example, it is a well-known fact that if a sphere of large permeability (an iron sphere, for example) be placed in a uniform field of intensity H, the resultant field at all points is such as would be obtained by superposing upon the original field of intensity H, the field of a magnetized sphere of very approximately the following properties when considered as *isolated from the original field H*.

Intensity of magnetization of sphere $= I = 3H/4\pi$

Field intensity within sphere $= -H$

Induction B_1 within sphere $= -H + 4\pi I = -H + 3H = 2H$.

The superposition of the field of this sphere upon the original field H gives a resultant intensity zero, and a resultant induction $3H$, inside the sphere, while outside the sphere the field is that composed of the field intensity H together with the field intensity due to the magnetized sphere. Now one who thought in terms of moving lines of force might naturally be tempted to regard the portion of the field due to the magnetized sphere, *i.e.*, the part corresponding to an induction $2H$ within the sphere, as rotating with it, and consequently producing no effect, and the remaining portion H as stationary and so giving rise to motional intensity in the rotating sphere. Such a view would result in a value of V only one third of that given by (1). Slightly varied views as to

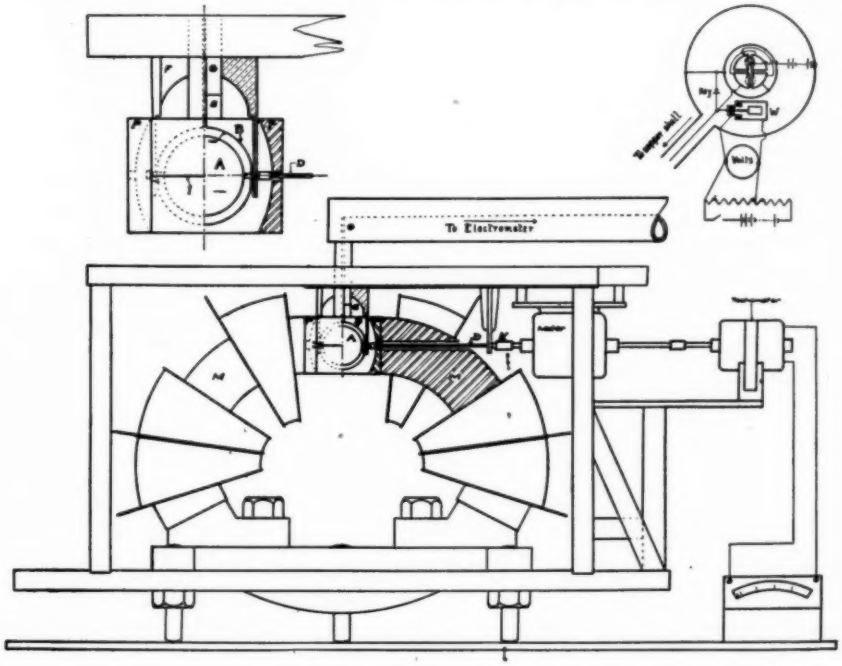

Fig. 1.

the phenomenon would lead to yet other conclusions. It would therefore seem that, at any rate from the point of view of a devotee of the moving line theory, something would be added to the conclusiveness of the arguments on which (1) is based by an experimental verification of that formula.

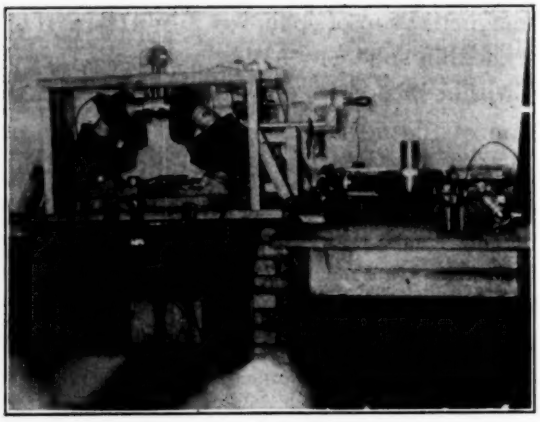

Fig. 2.

EXPERIMENTAL PROCEDURE.

General Description of Apparatus.—The apparatus is represented diagrammatically in Fig. 1, which is drawn to scale, and a photograph of the apparatus, exclusive of the electrometer system, is shown in Fig. 2. The iron sphere, and the parts immediately associated with it, were mounted between the poles of a large electromagnet *MM*. The sphere was driven through the shaft *D* by a high-speed motor, whose axle was connected at one end to a Hopkins electric tachometer, for the purpose of registering the speed of rotation. The pole pieces *PP* were specially formed so as to give a uniform induction within the sphere, in accordance with the tests to be described later. The rotating sphere *A* was surrounded by a thin copper shell *B*, made in two halves, and supported by a rod *G* passing thr ough a sulphur plug *S*, and connected to a Dolezalek electrometer as indicated. All parts of the insulated system were of course carefully inclosed in shields which were connected together electrically; and, the insulated shell itself was boxed in by a cylindrical zinc cover fastened to the brass frame *F* which supported the axle bearings. The bearings were of brass, and were very carefully made, on account of the high speed of rotation used. They were finally ground with rouge until the necessary perfection of fit was obtained, and were lubricated

[1] Thomson, J. J., Phil. Mag., Sept., 1915.

frequently while in use. The various parts concerned in the support of the sphere system, shaft, and motor, were mounted on a single stone slab for steadiness, and this slab was fitted into the solid wooden frame which supported the electromagnet, in the manner shown.

It was anticipated that charging of the copper shell might result from mere rotation of the iron sphere without magnetic field, as a consequence of friction with dust in the air. The inside of the copper shell was consequently smeared with vaseline, so as to insure that dust which struck it would stick, and in this way it was hoped that any undesirable effects of the kind anticipated would be quickly eliminated during the rotation of the sphere. That such effects were reduced to a sufficiently satisfactory minimum is borne out by the observations described below.

Contrary to expectation, it was found that the induction produced in the sphere by excitation of the electromagnet was always the same for a given exciting current in the coils, at least to the degree of accuracy with which the Grassot fluxmeter used would record. This was a matter of great convenience, because it enabled the fluxes to be determined once for all, for the various exciting currents used, so that all that was necessary in the main experiments was a measurement of the exciting current in the electromagnet. The flux, for the various exciting currents used, was measured by winding four turns of No. 36 double silk covered wire around that great circle of the iron sphere which was perpendicular to the axis of rotation, the leads from the coil being of course twisted together, and then taking the throw on the Grassot fluxmeter when the current was switched on, and when it was switched off. Considerable attention was given to the uniformity of the flux, a matter which will be discussed in detail later.

The electrometer system is shown at the top of Fig. 1, to the right, and is largely self-explanatory. The original intention was to measure the charge which it was necessary to give to the insulated shell to keep it at zero potential after the magnetic field had been applied to the rotating iron sphere; and, to this end, the insulated quadrant was connected to a small condenser W, whose outer member was connected to the sliding member of a rotary potentiometer, the potential applied being measured by a voltmeter. It was later found more convenient to read electrometer deflections; but, the condenser and potentiometer served a convenient purpose in determining the ratio q_{12}/q_{22}, and they were also of use in checking up the electrometer sensitivity at any time. For, if the factor to reduce electrometer deflection to volts be denoted by k, which is supposed determined by applying a known potential to the quadrant, if β is the deflection produced by raising the outer member of the con-

denser W from zero to U, and Q_{21} the coefficient of induction between the outer member in question and the insulated system, we have:

$$q_{22}k\beta + Q_{21}U = 0. \tag{2}$$

If on some future occasion, when k has changed slightly for some reason or other, we measure the corresponding values β_1 and U_1 of β and U, we have:

$$\frac{k_1}{k} = \frac{U_1\beta}{U\beta_1}, \tag{3}$$

which serves to determine k_1 in terms of k. Of course there is no special advantage in this method of checking up on the variations of k over that involved in measuring k itself on each occasion except that, the condenser W being part of the apparatus, no additional connections were necessary to enable a test to be made at any time.

General Procedure in the Main Experiments.—The sphere was set in rotation with an exciting current of 8 amperes in the electromagnet, and the speed of rotation was adjusted by an assistant to about 3,300 r.p.m., and kept there as steadily as possible, by means of a rheostat in the motor circuit. The same assistant also watched the electromagnet ammeter, and kept its reading at 8 amperes. When the conditions of speed had become steady, and under satisfactory control, the electrometer quadrant was released from earth. After any momentary fluctuations of the electrometer spot had subsided, and the reading was about to be taken, warning was given to the assistant, who then took special pains to keep the speed steady. A second or two later, the electrometer reading θ_1 was taken, and, at the same instant, the assistant recorded the speed, and ammeter reading. The current through the electromagnet was then reversed by means of a switch, while the sphere was kept in rotation. As a result, the electrometer showed a deflection. The assistant adjusted the electromagnet current and the speed to as nearly as possible the same values as before; and, when the electrometer spot had come to rest, he was again warned to prepare to read the ammeter and tachometer when the signal was given. A few seconds later, the electrometer reading θ_2 was taken, the assistant reading the speed and ammeter at the same instant. The current through the electromagnet was reversed once more, and readings corresponding to the first set were taken. The whole experiment was carried out, in the above manner, four times. The procedure was then repeated for exciting currents of 10, 12, and 16 amperes in the electromagnet. Finally, the whole series of observations for the different exciting currents was gone through again for speeds of rotation of 4,800 r.p.m. and 6,400 r.p.m.

In each of the above sets of observations, the agreement of the first and last electrometer readings θ_1 and θ_3 gave a good check on any possible charging effect which had resulted from causes other than that under investigation. The difference between θ_1 and θ_3, regardless of sign, and expressed as a percentage of the change of deflection on reversal of the magnetic field amounted, on the average, to 4.2 per cent. To the extent that the small spurious charging effect was uniform during the three readings, it of course became eliminated on taking the mean of $\theta_2 - \theta_1$ and $\theta_2 - \theta_3$; and, the potential change ΔV corresponding to this mean, since it is the change of *reversal* of the magnetic field, corresponds to twice the value of V given by (1), so that the theoretical formula for ΔV is:[1]

$$\Delta V = -\frac{4\pi Bna^2}{3c}\frac{q_{12}}{q_{22}} . \qquad (4)$$

A check on the sensitivity of the electrometer was made at the end of each day's observations, so that all that was necessary to complete the test of (4) was a determination of q_{12}/q_{22}.

Determination of q_{12}/q_{22}, and Electrometer Sensitivity.—We have seen (equation 2) that if β is the electrometer deflection resulting from the application of a potential U to the outer member of the condenser W,

$$q_{22}k\beta + Q_{21}U = 0.$$

If now we earth the outer member of W, insulate the iron sphere temporarily from its shaft and bearings, and, by means of a potentiometer, alter its potential from zero to v, the quadrant having been earthed and then insulated just before applying the potential v, we shall obtain a deflection of the electrometer. If u is the amount by which it is necessary to alter the potential of the outer member of W in order to compensate this deflection, we have:

$$q_{12}v + Q_{21}u = 0,$$

which, in combination with the previous equation, leads to:

$$\frac{q_{12}}{q_{22}} = \frac{k\beta u}{Uv} , \qquad (5)$$

so that the determination of q_{12}/q_{22} does not involve a knowledge of the absolute value of Q_{21}. In carrying out the second of the above experiments, the iron sphere was insulated from its bearings by a layer of paper.

[1] The fact that the earthed axle has a finite cross section and is not a geometrical line only produces an error of the order 0.5 per cent. in these experiments.

Corresponding to a value $U = 4.5$ volts, the value $\beta = 27.4$ divisions was obtained as a mean of three results with a maximum difference of one per cent. A simultaneous measurement of k gave $k = 1{,}226 \times 10^{-3}$ as a result of three determinations differing by less than one half of one per cent. As a mean of five almost identical determinations, the value 5.76 volts was found as the value of u necessary to compensate the potential $v = -0.333$ volt applied to the iron sphere, so that, inserting these values in (5), we obtain:

$$\frac{q_{12}}{q_{22}} = -0.1290.$$

Inserting this value in (4), together with the values 3.17 cms. for the radius of the iron sphere, and 3.00×10^{10} cm./sec. for the velocity of light, and introducing, moreover, a factor 1/60, so that in the resulting formula, n may be expressed in r.p.m., we have, for the theoretical value V of the change of potential of the shell as a result of reversing the magnetic field:

$$\Delta V = 0.906Bn \times 10^{-9} \text{ volt.} \qquad (6)$$

The final experiments were made on December 2, 4 and 6, 1920, and the values of k, determined in the manner described from the standard value $k = 1.226 \times 10^{-3}$ are respectively: 1.178×10^{-3}, 1.226×10^{-3} and 1.234×10^{-3} volt per millimeter.

Subsidiary Considerations.

Calculation of Mean Speed and Deflection.—As explained above, each experiment under any one set of conditions involves the observation of three electrometer readings, θ_1, θ_2, and θ_3, and three corresponding, but slightly differing speeds of rotation s_1, s_2, s_3. It might appear at first sight that the proper procedure in averaging the results would be to calculate $\Delta\theta = (\theta_2 - \theta_1 + \theta_2 - \theta_3)/2$ and take this as corresponding to the speed $(s_1 + s_2 + s_3)/3$. Such a procedure is not quite exact, however, for the reading θ_2 figures twice in the calculation of $\Delta\theta$, so that s_2 should be given more weight than would correspond to the above method. In fact, the matter may be regarded in the following light: If θ_0 is the true zero of the electrometer, we may write:

$$\theta_2 - \theta_1 + \theta_2 - \theta_3 = (\theta_0 - \theta_1) + (\theta_2 - \theta_0) + (\theta_2 - \theta_0) + (\theta_0 - \theta_3),$$

which shows that the mean value of $\Delta\theta$, calculated as above, is equivalent to that which would have been obtained if the mean of four readings of $\Delta\theta/2$ had been obtained, each one measured from the true zero of the electrometer, and the result had then been doubled. It is therefore

obvious that the appropriate mean speed to combine with the above mean value of $\Delta\theta$ is $(s_1 + s_2 + s_2 + s_3)/4$. In the results given in the final tables, the mean speeds have been calculated in this way.

The Uniformity of the Magnetic Induction, and the Calculation of B for the Final Experiments.—A sphere made from the same piece of iron as the main sphere, and of exactly the same size, was used for the tests on uniformity of the field. It was coated with wax; and, after it had been mounted on a lathe, three grooves were cut in the wax, one along a great circle, and the others along small circles parallel to the great circle. Coils of fine wire were wound in these grooves, and a fourth coil was wound around the axle. It was by comparing the throws obtained in the fluxmeter from these various coils when the magnetic field was turned off and on that the uniformity of the induction was investigated. We shall denote by subscripts 1, 2, 3, 4, the various coils, including the axle, subscript unity referring to the largest coil. In addition, we shall use the following symbols, with their appropriate subscripts, to designate the quantities indicated:

B—Average corrected [1] flux per sq. cm. through a coil.

N—Number of turns on coil ($N_1 = 4$, $N_2 = 6$, $N_3 = 20$, $N_4 = 20$).

S—Effective area of coil [1] ($S_1 = 31.9$, $S_2 = 21.05$, $S_3 = 6.70$ sq. cm.).

φ—Throw obtained in fluxmeter on turning on some definite current in the electromagnet.

α—Ratio of cross-sectional area of axle to that of coil. (The cross-sectional area of the axle was 0.173 sq. cm., and $\alpha_1 = 0.0054$, $\alpha_2 = 0.0082$, $\alpha_3 = 0.0252$.)

G—Constant for converting fluxmeter throws to e.m.u.

The fluxes through the various coils are then given by:

$$B_1S_1 = G\left(\frac{\varphi_1}{N_1} - \frac{\varphi_4}{N_4} + \alpha_1\frac{\varphi_1}{N_1}\right), \tag{7}$$

$$B_2S_2 = G\left(\frac{\varphi_2}{N_2} - \frac{\varphi_4}{N_4} + \alpha_2\frac{\varphi_2}{N_2}\right), \tag{8}$$

$$B_3S_3 = G\left(\frac{\varphi_3}{N_3} - \frac{\varphi_4}{N_4} + \alpha_3\frac{\varphi_3}{N_3}\right). \tag{9}$$

[1] Since the lines of induction enter the sphere practically normally, the area concerned is not the true mean area of the coil, but the mean area of the coil projection marked out on the sphere by dropping thereon perpendiculars from all points of the coil. The correction only amounted to one half of one per cent. for the thin wire used, except in the case of coil 3, where the employment of 20 turns of wire necessitated having more than one layer. In the case of this coil, the correction amounted to 2 per cent. The uncorrected areas were obtained by measuring the lengths of wire with which the various coils were wound.

Each side of one of these equations represents, of course, the average total flux through one turn of the corresponding coil.

These formulæ need some explanation. Taking the first formula, for example, the true average flux through one turn of the coil in question would be $G\varphi_1/N_1$. The axle contributes an unfair share to this flux, however, in view of the fact that it projects beyond the sphere, and so affords a particularly easy path through itself for the magnetic lines. Moreover, this excess flux through the axle contributes practically nothing to the effect in the main experiment, since it exists through a part of the sphere where the linear velocity of the material is practically zero. To allow it to contribute to the calculated flux would be equivalent to supposing that its effect was the same as if it were distributed uniformly throughout the sphere, whereas, its effect is quite negligible compared with what it would be were it so distributed. It is therefore desirable to omit, from the total flux, the excess contributed by the axle. If we subtract $G\varphi_4/N_4$ from $G\varphi_1/N_1$ we omit not the excess, but the whole flux through the axle, so that the result so obtained must be again corrected by the addition of $G\alpha_1\varphi_1/N_1$. Such then is the explanation of equations (7), (8), and (9). When it is observed that, in the case of the large coil, for example, the second two terms on the right-hand side of (7) form only about three per cent. of the first term, we see that the above method of procedure is quite justified.

Tests of the uniformity of the flux, *i.e.*, of the equality of B_1, B_2, and B_3 were made with currents of 8, 10, 12, and 16 amperes in the electromagnet coils. With flat poles, a very considerable lack of uniformity existed, and it was thought that the best results would be secured by mounting the iron sphere and its surrounding copper shell in a hole cut out of a cylinder of iron, which cylinder was fixed to the support F (Fig. 1), and fitted between the poles of the magnet. With this arrangement the effect is the same as if the magnetic circuit of the electromagnet were completed by iron, except for the spherical hole in which the copper shell and the iron sphere are mounted. Unfortunately, this plan was quite unsuccessful; nor was it possible, by pulling the poles away from the iron cylinder and inserting pieces of iron of various shapes, to improve the uniformity of the field to the desired extent. Finally, the iron cylinder was discarded, and the iron sphere, with its copper shell and zinc outer cover, were mounted directly between the poles of the electromagnet, the latter being cupped to the shape which gave the best results. With this arrangement, the following set of values was obtained for the flux per square centimeter through the various coils, with the electromagnet currents indicated:

TABLE I.

Showing Uniformity of Flux in Sphere.

Current in Magnet Amperes.	Flux per Sq. Cm. of Coil in E.M.U.		
	B_1.	B_2.	B_3.
8.....	3,980	4,030	3,760
10.....	4,660	4,720	4,470
12.....	5,190	5,260	5,020
16.....	6,120	6,250	6,010

The above numbers represent calculations, based on equations (7)–(9), from the mean of two sets of observations, each set being obtained by observing the mean of the fluxmeter throws on starting and stopping the current in the electromagnet. No two observations in corresponding sets differed by more than one per cent.; and, the agreement was, on the average, good to one half of one per cent. It must of course be borne in mind that the uncertainty of the results in the main experiments is less than the divergence from uniformity shown by Table I. For, although the average value of the flux per square centimeter through a great circle was used in the calculation of the final result, lack of uniformity of the field only introduces error to the extent that a transference of a portion of the flux from one part of a cross section to another does not leave the electromagnetic effect unchanged in view of the greater efficacy of the flux in the regions of larger radius, resulting from the greater velocity there. The comparatively good agreement between the values of B_1 and B_2, moreover, presents a very satisfactory feature, since it implies that the flux per square centimeter in the outer and more important regions of the sphere is closely equal to the *average* flux per square centimeter taken over a cross section of the great circle.

The calculation of the average flux per square centimeter through the great circle of the main sphere was not made on the basis of the above observations, but on the basis of observations made on the main sphere itself. As already stated, the observations consisted in measuring the throws obtained from a coil of four turns wound around the great circle. The effective area to use in the calculation of the flux per square centimeter is the area of the great circle, and not the mean area of the coil, since the field intensity just outside the sphere at the equator is sensibly zero. The difference between the two areas is of course extremely small. The values of B, obtained on the basis of equation (7), from two sets of observations in which no two determinations differed by more than one part in three hundred are: 6,070, 5,130, 4,610, and 3,930 e.m.u. for exciting currents of 16, 12, 10, and 8 amperes respectively.

THE FINAL RESULTS.

The complete sets of observations are too lengthy to print; and, it will suffice to give the condensed digest of the results shown in Table II.

TABLE II.

Abridged Table of Final Results for an Iron Sphere 3.17 Cms. in Radius.

Date.	No. of Obsns.	Speed R.P.M.	B E.M.U.	V (Volts $\times 10^{-8}$) Observed.	V (Volts $\times 10^{-8}$) Calculated.	Diff. Per Cent.
Dec. 2....	4	3,280	3,930	1.18	1.17	− 0.8
	4	3,450	4,610	1.44	1.44	0.0
	4	3,260	5,130	1.53	1.51	− 1.4
	5	3,250	6,070	1.73	1.79	+ 3.5
Dec. 4....	4	3,250	3,930	1.14	1.16	+ 1.8
	4	3,290	4,610	1.37	1.37	0.0
	4	3,300	5,130	1.54	1.53	− 0.7
	4	3,250	6,070	1.84	1.79	− 2.8
	4	4,830	3,930	1.80	1.72	− 4.4
	4	4,840	4,610	2.05	2.02	− 1.5
	4	4,850	5,130	2.28	2.25	− 1.3
	4	4,900	6,070	2.67	2.69	+ 0.7
	2	5,600	6,070	3.10	3.08	− 0.7
Dec. 6....	4	6,430	3,930	2.36	2.29	− 3.0
	5	6,460	4,610	2.78	2.70	− 2.9
	4	6,430	5,130	3.06	2.99	− 2.3
	4	6,430	6,070	3.59	3.53	− 1.7

The numbers in the second column indicate the numbers of sets of observations for which the data given represent the means. Thus, for example, the last line represents four observations in which the speeds were 6464, 6408, 6424, and 6432 r.p.m., giving the mean 6430 r.p.m., expressed to three figures, while the corresponding observed values of ΔV were 3.54, 3.60, 3.63, and 3.58 hundredths of a volt, giving the mean 3.59 hundredths of a volt. Moreover, each of these individual observations corresponds to three measurements of speed, and three readings of the electrometer, as explained above under "General Procedure in the Main Experiments." As stated, moreover, under that section, the discrepancy between the first and third readings of the electrometer was, on the average, only 4.2 per cent. of the change of deflection on reversal of the magnetic field; and, even this small amount became eliminated by the method of taking the observations. As an additional precaution, tests were made to ascertain whether any electrometer deflection was

produced by merely turning on the magnetic field in the absence of any rotation of the sphere; but, neither with the smallest nor with the largest field used was any vestige of such an effect observed. It is hardly necessary to add that the *sign* of the observed effect is in accordance with the theory, and corresponds to an increase in potential of the shell as a result of superposing a magnetic induction in such a direction that, as one looks in that direction, the direction of rotation of the sphere is clockwise.

The greatest difference shown by Table II., between the calculated and observed values of ΔV amounts to 4.4 per cent., and the average percentage difference, estimated regardless of sign, amounts to only 1.7 per cent.; so that, in view of the comparative smallness of the quantities measured, and of the rather exceptional conditions attending the employment of high speeds of rotation, and fairly large magnetic fields, the observations may be taken as a satisfactory verification of the theoretical formula.

The expenses of the above investigation have been, in part, defrayed by a grant made by the executive committee of the graduate school of the University of Minnesota. The writer desires to take this opportunity of expressing his sincere thanks to the committee. He also wishes to express his appreciation of the services rendered by his research assistants, Mr. W. B. Haliday, and Mr. D. Hartwell, and by the laboratory mechanician, Mr. C. Dane.

Department of Physics,
University of Minnesota,
August 29, 1921.

A NEW TONE GENERATOR.

BY C. W. HEWLETT.

SYNOPSIS.

New Pure Tone Generator and Receiver of Sounds.—(1) *Construction and operation.* The instrument consists of a thin, non-magnetic, metallic diaphragm between two flat coils through which a constant direct current I_0 flows in such a way as to produce a radial magnetic field in the diaphragm; then when a simple harmonic alternating current I of the frequency $\omega/2\pi$ is superposed upon the direct current, circular currents are induced in the diaphragm, which thereupon is acted upon by a simple harmonic electrodynamic force and vibrates with the frequency of the alternating current. For low frequencies the electrodynamic force is approximately proportional to $\omega I_0 I \sin(\omega t + \theta)$ and the amplitude of vibration is approximately proportional to $I_0 I/\omega$. The absence of overtones is due to the absence of ferromagnetic material, and to the fact that the radial magnetic field is constant. The aperiodicity of the diaphragm renders the calculation of the performance of the instrument practicable, and eliminates distorsion, due to resonance, in the wave form of the emitted sound when the instrument is excited by a complex alternating current. When used as a generator of pure tones, the coils were connected in the circuit of a thermionic oscillator whose frequency could be varied from 500 to 25,000 vibrations per second. When used as a receiver of sound, the current generated in the coils by the motion of the diaphragm is fed into a thermionic amplifier. (2) *Quantitative study of the performance.* The distribution of the magnetic field between the coils was determined experimentally; the diaphragm current equations were deduced and solved for a particular case; the forces on various parts of the diaphragm were calculated, and thence the amplitude of vibration and the sound energy output. With an aluminum diaphragm 0.0025 cm. thick and 10 cm. in diameter, a direct current of 1 ampere, an alternating current of 0.085 ampere, and a frequency of $10^6/2\pi$, these were respectively 7×10^{-7} cm., and 9 ergs per second. By increasing both direct and alternating currents five-fold, the output could be increased over six hundred-fold. Measurements of the amplitude for various frqeuencies agreed well with the calculated values. (3) *Applications of the instrument.* Since it gives a pure tone of constant and measurable pitch and intensity over a wide range, it would serve as a *precision source of sound*, useful both for research and lecture purposes. When used as a *telephone receiver and transmitter*, actual tests have shown that the reproduction of sound is remarkably faithful.

INTRODUCTION.

IN a previous publication[1] the author gave a brief description of a new instrument for generating simple harmonic sound waves. The following is a quantitative description and study of the instrument and its performance, together with a discussion of some of its applications.

[1] PHYS. REV., 17, p. 257, 1921.

I. Specifications for the Construction of the Tone Generator, and Description of a Method of Operation.

The instrument described below was used successfully at frequencies from 500 to 25,000 vib./sec. For lower frequencies the instrument should be wound with a larger number of turns, while for higher frequencies a smaller number of turns of coarser wire should be used.

Figure 1 is a section and elevation. The frame is made of hard fiber,

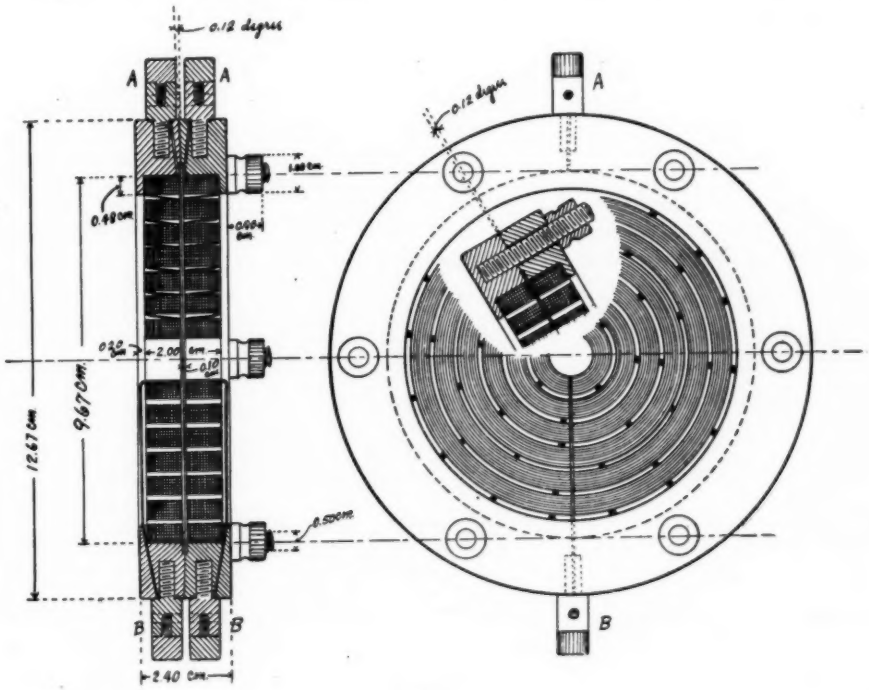

Fig. 1.

and the clamps and binding posts of brass. Each section of the pancake coils contains 98 turns of No. 22 wire, wound in 7 layers. The inside diameter of the innermost section is 1.11 cm., while that of the others is obtained by adding 1.27 cm. successively. These sections were wound on forms, shellaced, baked, and then bound at intervals with silk threads (not shown in figure). The sections mounted concentrically, are held in position by hard wood plugs. The sections of each coil are connected so that the direction of winding is continuous. The frame was turned after completing the coils, and the latter were glued in place with shellac. The taper on the clamping surfaces stretches the diaphragm enough to prevent appreciable sagging.

Any kind of metallic diaphragm will work; the author uses sheet aluminum 0.0025 cm. in thickness. Of the non-magnetic metals, alumi-

num gives the best results. An iron diaphragm gives more intense sounds at low frequencies for a given electrical input, although it will not emit a pure tone even when the instrument is excited by a simple harmonic alternating current. The diaphragm should be thin, as increasing the thickness decreases the intensity of sound emitted for a given electrical input, particularly for high frequencies. The reasons for the above may be drawn from the quantitative discussion of the performance of the instrument.

Figure 2 shows a method of using the instrument. The inductance

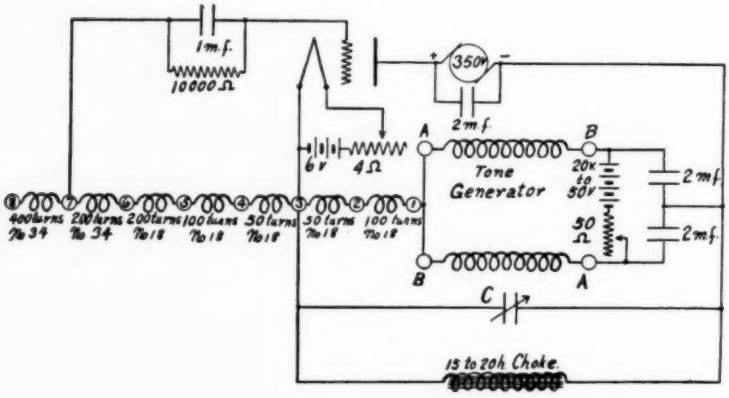

Fig. 2.

with the binding posts 1 to 8, is wound on a wooden core 20 cm. in diameter, and 6.5 cm. long. Coil 1 to 2 is innermost, and the others follow in numerical order. The wire is double cotton covered, and the layers are separated by friction tape. The best conditions for oscillation at any frequency can be obtained by adjustment of grid and filament connections to this inductance. In the actual circuit, *C*, and the choke coil are connected directly to the filament, and a single wire leads from this point to the inductance. The choke coil is home made. *C* consists of two 1 m.f. divided mica condensers, and five oil-filled rotary condensers whose combined maximum capacity is a little greater than the smallest step obtainable with the mica condensers. The other condensers are telephone condensers. The generator may be replaced by a battery, in which case no by-pass condenser is needed. The voltage shown is suitable for the Western Electric 205 B tube. The intensity of the sound from the instrument is controlled by the rheostat in series with the battery furnishing the polarizing current. The frequency of oscillation was measured with an improvised wave meter using a vacuum thermocouple for indicating resonance. If desired the instrument may be operated from a separate generator.

II. Quantitative Investigation of the Performance of the Instrument.

1. *Determination of the Distribution of the Magnetic Field in the Space occupied by the Diaphragm between the Pancake Coils.*—To simplify matters, the field of only one of the coils was investigated. Obviously the field is symmetrical about the coil axis. A flat spiral, of pitch 0.084 cm., consisting of 54 turns of No. 36 wire was wound in a shallow groove on the face of a fiber disc 10 cm. in diameter.

The spiral was connected to a ballistic galvanometer and a known current was established in the coil. The galvanometer deflection was determined for reversal of this current, (*a*) with the spiral and coil coaxially in contact, (*b*) when 1 mm. apart. The outside turns of the spiral were removed two at a time, making the above determinations after each removal, until only two turns remained. These determinations together with a calibration of the galvanometer yielded the necessary data for calculating the normal and radial components of the magnetic induction near the coil face. To simplify the calculation each spiral turn was assumed equivalent to a circular turn whose diameter is the mean diameter of the spiral turn.

Curve I., Fig. 3, shows the normal component of the magnetic induc-

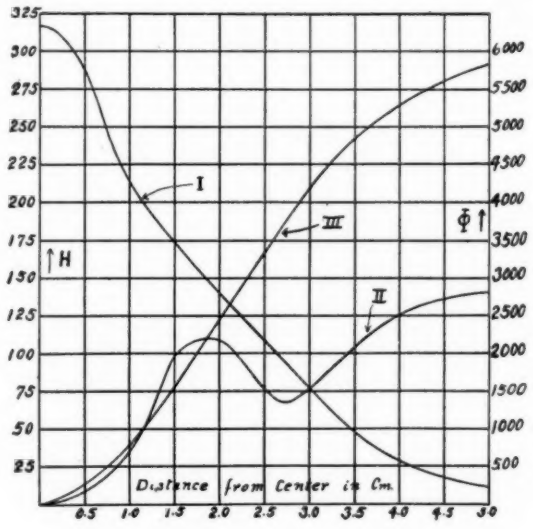

Fig. 3.

tion in C.G.S. units, as a function of the distance from the center, in the space between the two coils, when the latter are separated by a distance of 1 mm., and each carries a current of 1 ampere, their polarity being so adjusted that the unlike faces of the coils are adjacent. In this case

there is no radial component of the magnetic induction in the space between the two coils. Curve III. shows the total flux through a coaxial circle between the two coils as a function of the radius of the circle, for the same conditions of relative positions of the coils, current, and polarity as specified above. Curve II. shows the distribution of the radial component of the magnetic induction when the coils are separated by a distance of 1 mm., and each carries a current of 1 ampere, their polarity being so adjusted that the like faces of the coils are adjacent. In this case there is no normal component in the space between the two coils.

2. *Calculation of the Current Induced in, and Electromagnetic Forces Acting on the Diaphragm.*—The current in the diaphragm will flow in circles coaxial with the coils. The path of each current element will have a different resistance and self inductance, a different mutual inductance with the other paths, and will be acted upon by a different induced e.m.f. The current density and phase will therefore vary from one path to the next. The diaphragm was regarded as 8 annular rings each having a width of 0.645 cm., their mean diameters ranging from 0.97 cm. to 10 cm. Each annulus was considered equivalent to a circular wire of circular cross section of the same mass, material and diameter.

Suppose the current in the coils is given by $I_0 \sin \omega t$. The maximum values of the magnetic flux through each annulus was determined from curve III., and from these were calculated the induced e.m.f.'s in the annuli owing to the current in the coils. The total rise of potential around each annulus was then equated to 0, and the resulting eight equations put in the following form:

$$\left.\begin{aligned}
\left(\frac{r_1}{\omega^2} + a_{11}\right) i_1 + a_{12} i_2 + a_{13} i_3 + \cdots + a_{18} i_8 &= b_1 \sin \omega t + \frac{\Phi_1}{\omega} \cos \omega t \\
a_{21} i_1 + \left(\frac{r_2}{\omega^2} + a_{22}\right) i_2 + a_{23} i_3 + \cdots + a_{28} i_8 &= b_2 \sin \omega t + \frac{\Phi_2}{\omega} \cos \omega t \\
\cdots\cdots\cdots

The values of the coefficients were evaluated and the equations solved, giving the currents in the annuli, for a sine wave current whose maximum value in each coil was 0.085 amp., at a frequency given by $\omega = 10^5$. The current in an annulus may be expressed as follows $i_n = I_n \sin(\omega t + \theta_n)$. In the 4th and 5th columns of Table I. are given the values of I_n and θ_n for the successive annuli, beginning at the center. The second, third,

TABLE I.

I.	II.	III.	IV.	V.	VI.	VII.
n.	d_n Cm.	$r_n \times 10^3$ Ohms.	I_n Amperes.	θ_n Degrees.	F_n Dynes.	$A_n \times 10^7$ Cm.
1.....	0.97	5.67	1.71	56° 58′	4	0.30
2.....	2.26	12.1	2.33	43° 16′	73	2.36
3.....	3.55	20.7	2.81	41° 58′	345	7.10
4.....	4.84	28.4	3.11	40° 26′	388	5.85
5.....	6.13	35.8	3.32	38° 38′	511	6.08
6.....	7.42	43.3	2.89	43° 19′	768	7.55
7.....	8.71	50.8	2.77	43° 41′	1022	8.56
8.....	10.00	58.3	2.87	42° 52′	1263	9.22

and sixth columns of this table give respectively the diameters and resistances of the annuli, and the maximum values of the electromagnetic force acting on them. The latter are the products of the maximum current in each annulus, its circumference, and the corresponding radial component of the magnetic field for 1 amp. of direct current in each coil. The current in each annulus and the force acting on it are in phase, but the phases of the total instantaneous force and current differ slightly. These latter are given respectively by $f_t = 4423 \sin(\omega t + 41° 30')$, and $i_t = 22 \sin(\omega t + 42° 25')$.

3. *The Amplitude of Vibration of the Diaphragm, and the Amount of Sound Energy it Emits.*—From the above, the forces on the various parts of the diaphragm are nearly in phase; particularly so for those places where the force is large. Consequently, the diaphragm may be considered to vibrate as a whole. It is practically aperiodic for any frequency concerned, so that its elasticity may be neglected. The equation of motion is then

$$m\frac{d^2x}{dt^2} + a\frac{dx}{dt} = F \sin \omega t,$$

where m is the mass of the diaphragm, a is the dissipative factor, and F is the maximum periodic force. The solution of the equation gives for the amplitude of vibration,

$$A = \frac{F}{\omega \sqrt{a^2 + (m\omega^2)}}.$$

The dissipative factor a is made up of two parts; one due to internal friction in the diaphragm, and the other to the emission of sound energy. The former is readily seen to be negligible compared to $m\omega$, while the latter is equal to the product of the velocity of sound, the density of air, and the area of the diaphragm. For $\omega = 10^5$, a^2 is only 0.0025 of $(m\omega)^2$, and consequently it may be neglected. The amplitude then becomes $A = F/m\omega^2$. Substituting $F = 4423$ dynes, $m = 0.60$ gm. and $\omega = 10^5$, the amplitude comes out 7.37×10^{-7} cm. The sound energy emitted from both sides of the diaphragm per second is 9.1 ergs. By increasing both the direct current and the alternating current in the coils five-fold, which is feasible, the amplitude of vibration would be 25, and the energy output 625 times as great; *i.e.*, an output of 5.7×10^3 ergs/sec.

A better idea of the actual performance of the diaphragm is obtained by calculating the amplitudes of the separate annuli. Making the same assumptions as before these amplitudes were found and are given in the seventh column of Table I. An appendix to this paper gives some measurements of the amplitude at various places on the diaphragm.

From the current equations and the preceding discussion it is seen why a thin diaphragm should be used. For high frequencies r_n/ω^2 is small compared to a_{nn}, so that the currents in the diaphragm, and the forces acting on it are independent of its resistance. Increasing the thickness, therefore, does not appreciably increase the force, so that the increased mass results in a proportionally decreased amplitude of vibration. At low frequencies where r_n/ω^2 is of more importance, a thin diaphragm is still preferable, although the advantage is not so marked.

An iron diaphragm will not execute a simple harmonic motion with a sine wave current in the coils, for the magnetic flux through such a diaphragm will not be proportional to this current.

III. Some Applications of the Instrument.

1. *Precision Source of Sound.*—A practically pure tone can be obtained whose frequency and intensity can be maintained practically constant for an indefinite period, can be varied continuously over a very wide range, and can be determined with considerable accuracy. The practical absence of eddy current and magnetic hysteresis energy losses makes it possible to operate the instrument at very high frequencies. The natural period of the diaphragm is so low that it introduces no irregularities in the response in the region of operation.

Mr. C. E. Lane has recently used this instrument to determine the minimum flux of sound energy for audition at several frequencies, and it is intended that this work shall be carried further in the near future. In this connection it may be mentioned that the instrument may be constructed with one coil, so that one side of the diaphragm is free. This has the advantage that the waves are unbroken in leaving the instrument. The resulting tone is somewhat impure, because the radial component of the magnetic field, having an alternating component, is no longer constant. The impurities consist of the octave and higher partials of the fundamental, and can be eliminated by a sound filter. With a direct current large compared to the alternating current, these impurities are negligible. Above 10,000 vib./sec. the impurities are inaudible unless the fundamental is very intense.

The instrument is suitable for the demonstration of diffraction, interference, and reflection effects. At high frequencies, the sound goes out in a well-defined beam, which makes possible some very striking qualitative demonstrations.

For reproducing speech the instrument promises interesting development. It has been used successfully as a generator and as a receiver of voice currents. The coils were specially designed to suit each of these uses. As a generator, the vibration of the diaphragm induces alternating currents in the coils, and these currents are fed into a thermionic amplifier. The voice spoken into one instrument is reproduced with remarkable faithfulness in another connected in the plate circuit of the amplifier. This is due partly to the absence of eddy current and magnetic hysteresis energy losses, and partly to the aperiodicity of the diaphragm. As a loud-speaking generator of sound, the instrument has been used to address an audience of more than one hundred persons. The distinctness of the words, and the quality of the voice of the instrument approached very closely those of the speaker who operated the generator of voice currents.

For voice frequencies the current equations are much simplified. Only the terms containing r_n/ω^2 and the cosine terms need be retained, so that the diaphragm currents are approximately proportional to the frequency, and are nearly in phase. Consequently, the force acting on the diaphragm, for a given alternating current in the coils, is proportional to the frequency, and the amplitude of vibration is inversely proportional to the frequency. This would cause distortion if the voice currents were of the same wave shape as the sound waves which gave rise to them. But if the same kind of instrument is used as a generator, the wave form of its electromotive force is such that the distortion arising in the receiver results in the production of sound waves of the

same character as the original ones. As a matter of fact, actual trials have shown that the distortion, as perceived by the ear, even when using a carbon granule transmitter, is very small.

IV. APPENDIX. MEASUREMENTS OF THE AMPLITUDE OF VIBRATION OF THE DIAPHRAGM.

Mr. C. E. Lane, while a graduate student in this laboratory, conducted the experimental work described below. Using an electric micrometer for amplitude measurements, the amplitude of vibration at a definite frequency, of an ordinary telephone receiver, was first shown to be proportional to the current through the receiver, and then the ratio of amplitude to current was determined for several frequencies. The tone generator and telephone receiver were excited by alternating current of the same frequency, and a brass tube, of internal diameter 0.4 cm., leading to a tuned resonator carrying a Rayleigh disc, was placed close to the diaphragms, alternately of the tone generator and the telephone receiver. The current in the latter was adjusted till the deflection of the disc was the same for the two. The amplitude for the telephone receiver was calculated from the amplitude-current ratio, and it was assumed that this also gave the amplitude for the tone generator. In this manner the amplitude was determined at various places on the diaphragm, and its distribution was shown to agree closely with that given in Table I. The root mean square of the amplitude was calculated for the whole diaphragm and compared with the amplitude calculated as outlined in section II. Table II. shows the results of these measurements reduced to the standard condition of 1 ampere of direct current and 0.1 ampere of alternating current in the coils. The measured and calculated amplitude agree within the limits of experimental error.

TABLE II.

Frequency.	Measured Amplitude.	Calculated Amplitude.
1440	4.64×10^{-5} cm.	4.96×10^{-5} cm.
1600	4.16 "	4.52 "
2200	4.04 "	3.24 "
3700	1.72 "	1.88 "

The writer takes pleasure in acknowledging the assistance rendered by Mr. C. E. Lane and other students in this laboratory, in parts of this work. Credit is also due Mr. J. B. Dempster for suggestions in design and help in the construction of apparatus.

STATE UNIVERSITY OF IOWA,
IOWA CITY, IOWA,
October 5, 1921.

OMETER HAVING FLUXMETER CHARACTERISTICS.

By Paul E. Klopsteg.

Synopsis.

Deflection of an ideal fluxmeter in which the coil has great damping control but no torsional control, and has negligible inductance compared with its resistance, is shown to be directly proportional to the current and to the time during which it flows, for the range through which the magnetic torque is proportional to the current.

Measurement of time intervals with such a fluxmeter. In practice it would be necessary to calibrate the instrument, but the relation between deflection and time, for a constant current, should be approximately linear. A great advantage is that the range may be varied at will by proper control of the current.

SEVERAL years ago the writer published [1] the results of an investigation into the possibility of using a moving-coil galvanometer for the measurement of time intervals of the same order of magnitude as the half-period of the undamped galvanometer coil. The results of his experiments showed that such intervals, up to 6 seconds, could be measured with a galvanometer of 6.5 seconds' half-period, and that the probable error of a single observation did not exceed 0.5 per cent. In the work cited, results were obtained for two conditions of damping, viz., slightly damped and critically damped motion of the coil.

In the course of a more recent investigation, there was occasion to analyze the motion of a completely damped coil, *i.e.*, a coil without torsional control, as in the ideal fluxmeter, when the same conditions were imposed as have been outlined in the previous paper. The method of analysis may be briefly summarized as follows:

1. Set up the equation of motion of the coil, assuming zero torsional control.

2. From this equation, determine the angular position, θ_τ, and angular velocity, ω_τ, of the coil, initially at rest, at an instant τ seconds after a steady current has been started flowing in the coil.

3. Assuming that at the instant τ the current is interrupted, substitute the values θ_τ and ω_τ as initial conditions into the equation of motion of the coil, which, beginning at the instant τ, experiences no torque other than damping.

[1] Physical Review, VIII., 195, 1916.

4. Solve the resulting equation for the maximum angle of displacement, θ_m, in terms of τ.

Having obtained an expression involving the variables θ_m and τ—the other quantities being constant and determinable—the expression in the earlier work was made the basis for time measurement by observation of the angle θ_m corresponding to the unknown time interval, τ, during which a steady current passed through the galvanometer. In the present work we arrive at an expression which, on account of its simplicity, is interesting *per se*, and should be even better suited to time measurement than those applying to other conditions of damping, provided a galvanometer with characteristics approaching those of the ideal fluxmeter is used.

In the derivation, the major steps of which are reproduced below, the following symbols are employed, which are the same as those used in the writer's previous papers on galvanometers.[1] τ = time interval to be measured; θ = angular displacement of coil at time t; ω = angular velocity of coil at time t; θ_m = angle of maximum deflection; I_0 = moment of inertia of coil; $2f$ = proportionality constant between damping moment and angular velocity of coil; M = electromagnetic moment of coil; q^2 = elastic torque constant of suspensions; i = steady current impressed on circuit. Other designations are introduced as required. The principal assumptions to be made are that within the angle θ_m the magnetic field shall be uniform and radial, with resulting constant value for M; and that the inductive part of the reactance of the coil circuit shall be very small compared with the pure resistance, so that current growth and decay in this circuit are practically instantaneous.

Step 1.—The equation of motion, since the elastic torque constant $q^2 = 0$, is

$$\ddot{\theta} + \frac{2f}{I_0}\dot{\theta} = \frac{Mi}{I_0} \cdot \tag{1}$$

Step 2.—When $t = \tau$, this equation leads directly to

$$\theta_\tau = \frac{Mi}{2f}\left[\tau + \frac{I_0}{2f}\left(\epsilon^{-2f\tau/I_0} - 1\right)\right], \tag{2}$$

and

$$\omega_\tau = \dot{\theta}]_{t=\tau} = \frac{Mi}{2f}[1 - \epsilon^{-2f\tau/I_0}]. \tag{3}$$

Step 3.—At the instant τ, the current is interrupted, and the resultant orque on the coil vanishes. The general equation of motion then becomes

$$\ddot{\theta} + \frac{2f}{I_0}\dot{\theta} = 0, \tag{4}$$

[1] PHYS. REV., V., 266, 1915; VII., 633 and 640, 1916; VIII., 195, 1916; XV., 12, 1920.

which has the general solution

$$\theta = c_2 - \frac{c_1}{2a}\epsilon^{-2at}. \tag{5}$$

where c_1 and c_2 are integration constants, and $a = f/I_0$. To evaluate these, we impose the conditions: $t = 0$, $\theta = \theta_\tau$, $\dot{\theta} = \omega_\tau$, and find

$$\theta = \theta_\tau + \frac{I_0}{2f}\omega_\tau(1 - \epsilon^{-2ft/I_0}). \tag{6}$$

Step 4.—From inspection of equation (6) we see that θ has its greatest value θ_m when t is infinite, and that this value is given by

$$\theta_m = \theta_\tau + \frac{I_0}{2f}\omega_\tau. \tag{7}$$

Into equation (7) we introduce the values of θ_τ and ω_τ given in (2) and (3) and, upon simplifying, find

$$\theta_m = \frac{Mi}{2f}\tau. \tag{8}$$

Equation (8) shows that under the assumed conditions of an ideal fluxmeter, when used as an instrument for time measurement, the angle of deflection is accurately proportional to the time interval during which the current i is sent through the coil.

Under practical conditions, since the elastic torque of a fluxmeter suspension is never quite zero, such an instrument should be calibrated. This can readily be done by some such method as has been previously described.[1] The time scale will not depart far from uniformity, and its range can be varied at will by proper control of the current.

CENTRAL SCIENTIFIC COMPANY,
CHICAGO, ILL.,
August 15, 1921.

[1] Loc. cit., p. 204.

THE EFFECT OF DAMPING ON THE WIDTH OF X-RAY SPECTRUM LINES.

BY G. E. M. JAUNCEY.

SYNOPSIS.

Width of X-ray Spectrum Lines Due to Damping.—(1) Assuming exponential damping, the equation for the ether vibrations, $y = e^{-kt} \sin(pt + A)$, is developed as a Fourier integral and thus a *theoretical relation of the width to the damping factor* is found. If we define the width of the line as the range in wave-length throughout which the intensity is greater than half the maximum intensity of the line, then $x\lambda/\lambda = dp/p = 2k/p$, very closely.

(2) *Theoretical Variation with Wave-length.* Assuming in addition that k has the value given by the classical theory of a single radiating electron, $d\lambda = 2k\lambda/p = 4\pi e^2/3mc^2 = 1.2 \times 10^{-4}A$, *i.e.*, $d\lambda$ is independent of wave-length.

(3) *Comparison with Observation.*—A. H. Compton has found the widths of the W lines, $\lambda\lambda$ 1.242 and 1.279A, to be about $9 \times 10^{-4}A$, nearly eight times the above value. The discrepancy indicates that either the theory is incorrect, or the measurement is wrong, or that damping is not the chief cause of the observed width.

IN this issue of the PHYSICAL REVIEW, Professor A. H. Compton describes experiments in which he has measured the width of x-ray spectrum lines. The lines used were the tungsten lines $\lambda = 1.242$ and $\lambda = 1.279$ Å.U. After correcting for the angular aperture of the slits, the angular faults of the crystal grating, and the finite resolving power of the grating, Professor Compton finds that the width due to the non-homogeneity of the x-rays is given by $(\delta\lambda/\lambda) > 0.0007 \pm .00014$, or a width of the order of, but greater than, 0.5 minute of arc in the first order spectrum as reflected from calcite. It is the purpose of this paper to examine into a possible explanation of this width on the assumption that the wave motion sent out by the oscillator emitting x-rays can be represented by an equation of the type $y = e^{-kt} \sin[p(t - x/c) + A]$. This damped harmonic motion can be represented by a Fourier's integral. We shall assume that the oscillations begin at time $t = 0$, so that $y = F(t) = 0$ for $0 > t > -\infty$, but that $y = F(t) = e^{-kt} \sin(pt + A)$ for $\infty \geqq t \geqq 0$.

The formula for the expression of $F(t)$ as a Fourier's integral is well known[1] and we obtain

[1] Cf. Byerly, Fourier Series and Spherical Harmonics, p. 54.

$$e^{-kt}\sin(pt+A) = \frac{1}{\pi}\int_0^\infty \frac{d\omega\cdot\sqrt{C^2+D^2}\cdot\cos(\omega t-\tan^{-1}D/C)}{(p^2-\omega^2+k^2)^2+4k^2\omega^2},$$

where

$$C = p(p^2-\omega^2+k^2)\cos A + k(p^2+\omega^2+k^2)\sin A,$$

and

$$D = 2kp\omega\cos A - \omega(p^2-\omega^2-k^2)\sin A.$$

If now I_ω represents the intensity of the radiation of frequencies between

$$\frac{\omega-(\delta\omega/2)}{2\pi} \text{ and } \frac{\omega+(\delta\omega/2)}{2\pi},$$

we have

$$(1)\quad \frac{I_\omega}{I_p} = \frac{\omega^2}{p^2}\times\frac{(p\cos A+k\sin A)^2+\omega^2\sin^2 A}{p^2+2kp\cos A\sin A+k^2\sin^2 A}\times\frac{k^2(4p^2+k^2)}{(p^2-\omega^2+k^2)^2+4k^2\omega^2},$$

where I_p is the intensity when $\omega = p$. In Professor Compton's experiments the intensities were plotted against the wave-lengths and the difference between the two values of λ for which the intensity was one half of that of the center of the line was taken as $\delta\lambda$. We shall therefore take $I_\omega/I_p = \frac{1}{2}$, and

$$\frac{\omega-p}{p} = \frac{1}{2}\frac{\delta\lambda}{\lambda} = .00035 = a.$$

Since ω is very nearly equal to p and since therefore $p^2\cos^2 A+\omega^2\sin^2 A$ is very nearly equal to p^2, equation (1) reduces to the approximate form

$$(2)\quad \frac{I_\omega}{I_p} = b = \frac{k^2(4p^2+k^2)}{(p^2-\omega^2+k^2)^2+4k^2\omega^2},$$

which is independent of the phase angle A. Solving this equation for k and putting $\omega = p(1+a)$ where a is small, we obtain approximately

$$(3)\quad k = \frac{2\pi ac}{\lambda}\sqrt{\frac{b}{1-b}},$$

where $p = 2\pi c/\lambda$ and c is the velocity of light in vacuo. If b is taken as $\frac{1}{2}$, $k = pa = \omega - p$.

In Professor Compton's experiments $a = .00035$, $b = \frac{1}{2}$, and $\lambda = 1.242$ Å.U. for one of the lines observed. These values give $k = 5.3\times 10^{15}$ sec^{-1}. The amplitude of the vibration $y = e^{-kt}\sin(pt+A)$ is therefore damped to $1/e$ of its initial value in 1.89×10^{-16} secs. In this time the oscillator makes 457 vibrations and the energy of the wave train given

out by the oscillator therefore becomes damped to $1/e$ of its value in 229 wave-lengths.

Professor Compton gives the damping factor due to radiation for a single electron on the classical electrical theory as being equal to

$$\frac{4\pi^2}{3} \cdot \frac{e^2}{m\lambda^2 c}. \tag{4}$$

For $\lambda = 1.242$ A.U. this gives $k = 7.2 \times 10^{14}$ sec^{-1}, the amplitude being damped to $1/e$ of its value in 3,360 vibrations. Assuming that the damping factor of the emitted x-rays is equal to that of the oscillator, we have from (3) that a, or $\delta\lambda/2\lambda$, is equal to

$$\frac{k\lambda}{2\pi c}\sqrt{\frac{1-b}{b}} = .000047. \tag{5}$$

This corresponds to a width of 4 seconds of arc in the first order spectrum as reflected from calcite, the experimental value being greater than 30 seconds of arc.

The inconsistency between the damping factor as calculated by means of Fourier's integral from the observed width and that as calculated from formula (4) may be due to, (*a*) the incorrectness of the formula (4) as applied to the x-rays emitted by the oscillator, (*b*) the inaccuracies of the experiments measuring the width, (*c*) the possibility that the spectrum lines are made up of a finite number of lines close together, and (*d*) the possibility that the oscillation at a point in the ether due to the emitted x-rays cannot be represented by an equation of the form $y = e^{-kt}\sin(pt + A)$. It is hoped in the near future to make a redetermination of the width experimentally so as to eliminate (*b*).

The damping of the oscillator which emits any electromagnetic radiation must tend to broaden the spectrum line observed. This is true for light waves as well as for x-rays. From formula (4) it is seen that for light waves

$$a = \frac{\delta\lambda}{2\lambda} = 1 \times 10^{-8}$$

approximately. The broadening of the spectrum lines for ordinary light due to damping is therefore too small to be observed. In the x-ray case, however, an appreciable part of the width (about 13 per cent. according to Professor Compton's experiments) is due to damping. It is interesting to note that from the formulas (4) and (5) $a = \delta\lambda/2\lambda$ varies as $1/\lambda$, so that $\delta\lambda$ is constant for all values of λ throughout the x-ray

and light spectra.[1] This value of $\delta\lambda$ due to damping equals 1.2×10^{-4} Å.U. approximately.

The writer's thanks are due to Professor Compton of Washington University for his interest in the preparation of this paper.

Physics Laboratory,
Washington University,
St. Louis,
June 8, 1921.

[1] The writer is informed that Mr. C. G. Darwin, of Cambridge University, England, has also noted this interesting fact.

THE WIDTH OF X-RAY SPECTRUM LINES.

BY ARTHUR H. COMPTON.

SYNOPSIS.

Width of X-ray Spectrum Lines.—(1) *Four causes* are discussed: (*a*) the width of the slit employed, (*b*) angular imperfections of the crystal, (*c*) finite resolving power of the crystal grating, and (*d*) lack of homogeneity of the x-rays. The effect of the first two causes is independent of the angle of reflection and therefore the same for all orders and all lines; the effect of the third is shown to vary as $1/\sin\theta\cos\theta$ and therefore to decrease as θ increases up to 45°; while the effect of the last is equal to $(d\lambda/\lambda)\tan\theta$ and therefore increases with θ. (2) *Measurements of two tungsten lines*, λ 1.242 and 1.279 Å, in the first four orders from the cleavage faces of calcite and rock-salt, are given. The fact that in both cases the width is least for the second order and greatest for the fourth order shows that both the third and fourth causes contribute measurably to the observed width, as well as the first two.

Lack of homogeneity of tungsten x-ray lines computed from the above measurements comes out greater than $0.0007\lambda \pm .00013\lambda$, which is about 0.5′ for the first order. This cannot be explained as a Doppler effect or as due to the damping of the electronic motion; however it is in accord with the complexity of the lines as predicted by Sommerfeld's theory of elliptic electronic orbits.

SEVERAL years ago the writer performed a series of experiments designed primarily to measure the relative intensities of the different orders of x-ray spectrum lines reflected from crystals of rock-salt and calcite. A recording x-ray spectrometer, which has previously been described,[1] was employed, the intensities of the lines being taken proportional to the area under the curves on the record representing the different spectrum lines. These intensity measurements, in which cleavage faces of the crystals were employed, were later found to be valueless as a test of the theory of reflection, because of the selective absorption which occurs at the angle of maximum reflection from a cleavage face.[2] The results show, however, an interesting broadening of the lines in the higher orders.

The width of the spectrum lines was measured at a point midway between the peak of the curve and a base-line representing the intensity of the general radiation. The average values thus found for the tungsten lines $\lambda = 1.242$ and $\lambda = 1.279$ Å.U. are as follows:

[1] A. H. Compton, PHYS. REV., 7, 658 (1916).

[2] A. H. Compton, PHYS. REV., 10, 95 (1917); W. L. Bragg, James and Bonsanquet, Phil. Mag., 41, 309 (1921).

TABLE I.

Width in Minutes of Arc at Mid-point of Line.

Crystal.	First Order.	Second Order.	Third Order.	Fourth Order.
Calcite	16.3 ± .6	16.0 ± .36	17.3 ± .7	18.3 ± .12
(Calc.)	15.4	(16.0)	16.7	(18.3)
Rock-salt	20.1 ± .5	18.9 ± .3	20.3 ± .6	21.2 ± 1.2
(Calc.)	18.3	(18.9)	19.8	21.5

The size of the slits remained unchanged throughout the experiments. The measurements on the second and fourth orders of reflection from calcite are the most valuable, because these two lines are of nearly equal intensity, which tends to eliminate any consistent errors.

The observed width of a spectrum line may result from four different causes, (*a*) the angular aperture of the slits, (*b*) angular faults in the crystal grating, (*c*) the finite resolving power of the grating, and (*d*) the lack of homogeneity of the incident x-rays.[1] The width due to the size of the slits will obviously be the same at all angles. That due to angular faults in the crystal may vary somewhat according to the part of the crystal which is exposed to the x-rays, but should on the average be the same for all angles of reflection. It is clear that no crystal will act as a perfect grating, and that the resolving power will depend upon the size of the portions of the crystal which are effectively perfect. It can be shown[2] that if these perfect portions are assumed to be parallelopipeds of width along the crystal face δx, height δy and depth into the crystal δz, the effective width of the line (area/height) in radians is:

$$(1) \qquad \delta\theta_c = \frac{\lambda}{2\delta x \sin\theta} \quad \text{for} \quad \frac{\delta x}{\delta z}\tan\theta < 1,$$

$$\delta\theta_c = \frac{\lambda}{2\delta z \cos\theta} \quad \text{for} \quad \frac{\delta x}{\delta z}\tan\theta > 1,$$

where λ is the wave-length and θ the glancing angle. In crystals of rock-salt and calcite δx and δz are doubtless on the average about equal, so that $\delta\theta_c$ will be a minimum for $\theta = \pi/4$, increasing symmetrically for smaller and larger angles. Because of chance variations in δx and δz

[1] In the experiments here described, the beam incident upon the crystal was collimated by a pair of narrow slits, and the opening at the ionization chamber was wide enough to admit all the reflected rays. With Bragg's method of using narrow slits near the x-ray tube and at the ionization chamber and a wide slit at the crystal, the width also depends upon a fifth factor, the penetration of the x-rays into the crystal.

[2] This follows (after considerable reduction) from the results given by the writer in THE PHYSICAL REVIEW, 9, 37 (1917).

for the different component crystals, the change from $\delta\theta_c \propto 1/\sin\theta$ to $\delta\theta_c \propto 1/\cos\theta$ will be gradual. Thus it appears that the width of the line due to this finite resolving power should vary approximately according to the expression

$$\delta\theta_c \propto 1/\sin\theta\cos\theta. \tag{1a}$$

The angular width due to a given range of wave-lengths $\delta\lambda$ in the incident x-rays may be found by differentiating $\lambda = 2D\sin\theta/n$ with respect to θ. Thus we find that

$$\delta\theta_d = \frac{n\delta\lambda}{2D\cos\theta} = \frac{\delta\lambda}{\lambda}\tan\theta. \tag{2}$$

The total width of the line is a rather complicated function of $\delta\theta_a$ due to the width of the slits, $\delta\theta_b$ due to angular imperfections of the crystals, and $\delta\theta_c$ and $\delta\theta_d$ given by equations (1) and (2) respectively. Thus we may write

$$\delta\theta = F(\delta\theta_a, \delta\theta_b, \delta\theta_c, \delta\theta_d).$$

The function F will increase with every increase in any variable $\delta\theta_r$, in such a manner that $\delta\theta$ will be greater than any one of its components but will be less than the sum of all four. The form of this function cannot be determined unless we know the exact manner in which the intensity varies with the angular distance from the center of the line on account of each factor contributing to the width. Lacking such definite knowledge, a simple graphical analysis shows that an increase in $\delta\theta_r$ will result in an increase in $\delta\theta$ of the same order of magnitude but smaller than the change in $\delta\theta_r$, *i.e.*,

$$1 > \frac{\partial\delta\theta}{\partial\delta\theta_r} > 0.$$

Furthermore by Taylor's theorem, to a first approximation,

$$\delta\theta = \delta\theta_0 + \frac{\partial\delta\theta}{\partial\delta\theta_a}\Delta\delta\theta_a + \frac{\partial\delta\theta}{\partial\delta\theta_b}\Delta\delta\theta_b + \frac{\partial\delta\theta}{\partial\delta\theta_c}\Delta\delta\theta_c + \frac{\partial\delta\theta}{\partial\delta\theta_d}\Delta\delta\theta_d, \tag{3}$$

where $\Delta\delta\theta_r$ represents a change in $\delta\theta_r$.

In the experiments using a calcite crystal, $\delta\theta_a$ and $\delta\theta_b$ remain constant, $\delta\theta_c$ decreases with θ (up to 45°), and $\delta\theta_d$ increases with θ. The observed increase in the width of the lines in the higher orders therefore means that $(\partial\delta\theta/\partial\delta\theta_d)\Delta\delta\theta_d$ is more prominent than $(\partial\delta\theta/\partial\delta\theta_c)\Delta\delta\theta_c$. If we neglect the term in $\delta\theta_c$ and compare the value of $\delta\theta$ at two angles θ_1 and θ_2, it follows from expression (3) that

$$\delta\theta_2 - \delta\theta_1 = \frac{\partial\delta\theta}{\partial\delta\theta_d}(\delta\theta_{d_2} - \delta\theta_{d_1})$$

whence according to equation (2)

$$\frac{\delta\lambda}{\lambda} = \frac{\partial\delta\theta_d}{\partial\delta\theta} \frac{\delta\theta_2 - \delta\theta_1}{\tan\theta_2 - \tan\theta_1}. \tag{4}$$

Substituting the values of $\delta\theta$ observed in the second and fourth orders from calcite, noting that the angles θ_2 and θ_4 are 24.2° and 55.2° respectively and that $\partial\delta\theta_d/\partial\delta\theta$ is of the order of but greater than unity, we find that $\delta\lambda/\lambda$ is of the order of but greater than $0.0007 \pm .00014$. The values calculated in the second row of Table I. result from the further approximation that $\partial\delta\theta_d/\partial\delta\theta$ is constant for all values of $\delta\theta_d$. If we take this constant as unity, we find 0.0007 for the minimum value of $\delta\lambda/\lambda$, and 0.5′ for the minimum value of the width of the first order spectrum line due solely to lack of homogeneity of the x-rays. The observations on rock salt lead to values of the same order of magnitude.

The widths calculated for the lines reflected from calcite agree well with the experiments except in the first order. This line appears to be too broad, as we have noticed should result from the small resolving power if the crystal is sufficiently imperfect. The imperfection of the rock-salt crystal as compared with calcite shows itself both in the greater value of the constant $\delta\theta_b$ for rock salt, indicated by a greater width in all orders, and by the greater difference between the breadth of the first and second orders. According to equation (1*a*) the broadening due to this imperfection is nearly proportional to $1/\sin\theta\cos\theta$, *i.e.*, to 2.5, 1.3, 1.0 and 1.1 in orders 1, 2, 3 and 4 respectively. A difference in breadth of 1.8′ between the first and second orders of rock salt[1] will therefore correspond to a difference of only 0.3′ between the second and fourth orders. The unmistakable broadening of the higher orders can apparently be accounted for only by a true lack of homogeneity of the x-ray spectrum line. Thus it appears that all the terms of expression (3) have values that are appreciable in experiment.

It is of interest to consider the possible origin of the observed non-homogeneity of the x-rays. Let us consider first the width to be expected on the basis of the Doppler effect. The average velocity of the tungsten molecules in the target at 2000° K. is only about 10^{-6} times the velocity of light, and could therefore account for only a negligible part of the non-homogeneity of the x-rays. The velocity of the thermal motion of an electron, according to the principle of equipartition of energy, would give rise to a Doppler effect of the required order of magnitude; but the fact that the spectrum lines are characteristic of the tungsten

[1] This difference corresponds to a value of $\delta\theta_e > 3.2'$ in the first order. Thus by equation (1), $\delta x < c.\ 2700\lambda = 4.5 \times 10^{-5}$ cm.

atom indicates that it is not a free electron which emits the radiation, but one which is a part of the atom.

The damping of the electron's linear oscillations due to its own radiation, on the other hand, results in a width which is not negligible. The damping factor for a single oscillating electron is

$$\frac{4\pi^2}{3}\frac{e^2}{m\lambda^2 c},$$

where e, m and C have their usual significance. For $\lambda = 1.25$ Å.U., this means that the wave will be damped to $1/e$ of its amplitude in 3,400 vibrations. By an application of Fourier's integral, Professor Jauncey finds[1] that the effective width of the line due to this damping is about 0.06 minute of arc in the first order, which is 12 per cent. of the minimum width found experimentally.

It appears probable that the chief cause of the non-homogeneity observed in these experiments is that the lines themselves are complex. Thus the line $\lambda = 1.279$ is known to have a faint companion 0.008 Å.U. distant which was not separated in the first and second orders by the slits employed; and according to Sommerfeld's theory of the fine structure of spectrum lines, the line 1.242, on which most of the fourth order measurements were made, should have faint satellites at distances of about 0.0007, 0.002 and 0.006 Å.U.

The experiments described in this paper were performed at Palmer Physical Laboratory, Princeton University.

WASHINGTON UNIVERSITY,
SAINT LOUIS,
July 18, 1921.

[1] Cf. G. E. M. Jauncey elsewhere in this number of the PHYSICAL REVIEW.